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Minimum Principles in Plasticity

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Minimum principles in plasticity

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Zusammenfassung

Für eine Klasse von elasto-viskoplastischem Material wird das Randwertproblem mit Methoden der konvexen Analysis beschrieben. Ein allgemeines Minimalprinzip wird aufgestellt. Die Lösung wird sowohl für das Anfangs-Randwertproblem als auch für das Zuwachs-Randwertproblem konstruiert. Die Eindeutigkeit der Spannung als Funktion der Zeit wird bewiesen. Eine spezielle Klasse regulären Materials, für welche die Dehnung als Funktion der Zeit eindeutig ist, wird eingeführt. Ein Beispiel der Anwendung des Minimalprinzips zur numerischen Lösung des Problems wird vorgestellt.

Summary

Convex analysis approach to the boundary value problem for a class of elastic-viscoplastic materials is given. A general form of the minimum principle is established. The solution is constructed both for the initial-boundary value problem and for the rate boundary value problem. The uniqueness of stress history is proved. A particular class of regular materials, for which the strain history is unique, is introduced. An example of application of the minimum principle to a numerical solution of the problem is presented.

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P. Rafalski

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CONTENTS

1. INTRODUCTION

The present work is an attempt to give a uniform mathematical approach to the elastic-viscoplastic boundary value problem. Here we assume small deformation of the body and we confine our considerations to the generalized standard material model, formulated by NGUYEN [8] and HALPHEN [10]. The model is based on the concept of the generalized strain and generalized stress introduced by NGUYEN in [8] and on the hypothesis of normal dissipation postulated by MOREAU [5] and ZIEGLER [3].

Making use of assumed existence and convexity of the plastic potential in the generalized stress space we formulate the plastic flow law in terms of fundamental concepts of convex analysis, using the approach presented by EKELAND and TEMAM in [13].

Taking advantage of the assumed linear relation between the generalized elastic strain and generalized stress we construct the Hilbert space of admissible fields in which the solution is to be found. The principles of such construction are described by MAURIN in [4]. The concept of finitely-valued function is presented by YOSIDA in [14].

We solve parallely the primary initial-boundary value problem and the simplified rate boundary value problem using the methods of convex analysis in the Hilbert space. The minimum principles for stress derived here follow directly from the construction of solution in the Hilbert space. In the particular cases we obtain the minimum principles already existing in literature.

Since the fundamental works of HILL [1,2], where he established the first variational principles in plasticity, the important results obtained in this field concern the rate boundary value problem. The Hill's principle has been generalized by MRÓZ and RANIECKI [9,11] for the thermoplastic boundary value problem. The results of MOREAU [5] and MANDEL [6] concerning the existence and uniqueness of solution for the elastic-perfectly plastic body had been extended by NGUYEN [8], FRELAT and ZARKA [12] onto the case of generalized standard material.

The minimum principles corresponding to the initial-boundary problem for the generalized standard material have been established by RAFALSKI in [15,16,17,20,21]. The functional constructed there depends on the history of plastic deformation. The uniqueness of the stress field in the space-time region is proved. The regular material model, which assures the uniqueness of strain field, is introduced.

Other attempts to include the history of plastic deformation into a minimum principle can be observed in works of HALPHEN and NGUYEN [9] and NAYROLES [18,19].

An application of the minimum principles to the numerical calculations has been described by NGUYEN in [22]. The existing algorithms are based on the principle for the rate problem. Consequently one uses step-by-step numerical technique based on the finite element geometrical idealization. To improve the convergence of solution the implicit scheme of calculation was introduced. It was shown by NGUYEN in [22] that the particular implicit scheme proposed by MOREAU assures the convergence and stability of solution. This scheme reduces the problem to the convex programming in every time step.

It is shown in the present work that the application of global minimum principle, corresponding to the initial-boundary value problem, leads in the particular case to the implicit scheme proposed by MOREAU in [7].

1.1 Notation

In the present work vectors and tensors are denoted either by a symbol with subscripts t_{ij} or by the same bold type symbol without subscripts $\underset{\sim}{t}$. In general subscripts i, j, k, l run over integers 1, 2, 3. For these subscripts the summation convention is used, e.g. $\underset{\sim}{f} \cdot \underset{\sim}{u} = f_i u_i = f_1 u_1 + f_2 u_2 + f_3 u_3$.

Differentiation with respect to Cartesian coordinates $\underset{\sim}{x} = x_i$ is indicated by subscript i preceded by the comma. Differentiation with respect to time t is indicated by the dot above the symbol.

Throughout the work we shall distinguish between the tensor $\underset{\sim}{t}$ and the tensor field $\underset{\sim}{t}$. In the case of possible confusion we shall indicate the domain of definition of the field $\underset{\sim}{t}$ writing explicitly its arguments $\underset{\sim}{t}(\underset{\sim}{x},t)$.

2. BASIC ASSUMPTIONS

We shall consider three-dimensional bounded region V in three-dimensional space R^3. We assume that the boundary B of the region V is composed of surfaces, which have the normal unit vector $\underset{\sim}{n}$ taken as positive outwardly. We shall assume that the boundary B consists of the part B_k, where the displacement $\underset{\sim}{p}$ is prescribed and the part B_s, where the boundary force $\underset{\sim}{f}$ is prescribed. The body force $\underset{\sim}{b}$ is prescribed in the volume V in such a way that the entire body is in equilibrium.

The boundary value problem is represented by

Equilibrium equation	$\sigma_{ij,j} = b_i$	in V	
Boundary force condition	$\sigma_{ij}n_j = f_i$	on B_s	
Compatibility equation	$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$	in V	(2.1)
Boundary displacement condition	$u_i = p_i$	on B_k	
Constitutive relations describing the elastic-plastic behaviour		in V	

which should be satisfied for every $t_o > 0$ together with

Initial conditions	$\sigma_{ij} = 0$ and $\varepsilon_{ij} = 0$ at $t = 0$	in V	(2.2)

where $\underset{\sim}{\sigma}$ is the stress tensor field, $\underset{\sim}{\varepsilon}$ is the total strain tensor field and $\underset{\sim}{u}$ is the displacement vector field defined in the space-time region $\underset{\sim}{V} = V \times [0,\infty)$.

The three-dimensional boundary $\underset{\sim}{B} = B \times [0,\infty)$ is, in general case, composed of two arbitrary disjoint, sufficiently regular space-time parts $\underset{\sim}{B}_s$ and $\underset{\sim}{B}_k$

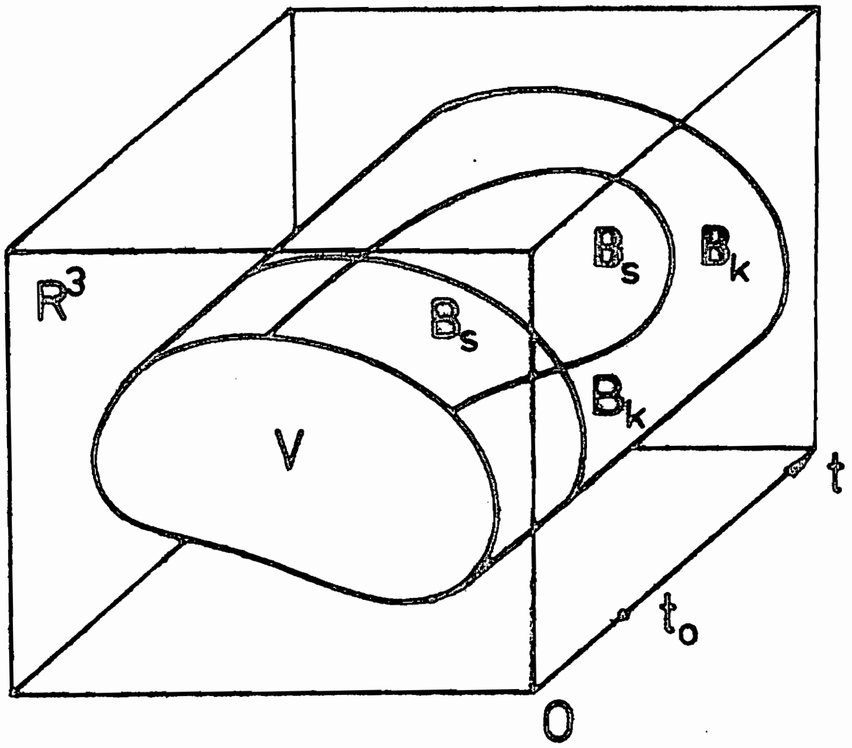

Fig. 1. Four-dimensional space-time region

The derivatives appearing in the equilibrium equation and the compatibility equation have the generalized sense, as we assume neither differentiability nor continuity of the stress or displacement fields. The physical interpretation of those equations follows from the primary concepts of equilibrium and compatibility.

Namely, the equilibrium equation in generalized sense is equivalent to the requirement of <u>force equilibrium</u> for arbitrarily chosen, sufficiently regular three-dimensional subregion V_n of the region V

$$\int_{B_n} \sigma_{ij} n_j \, d\underset{\sim}{x} = \int_{V_n} b_i \, d\underset{\sim}{x} \quad \text{for every } V_n \subset V \qquad (2.3)$$

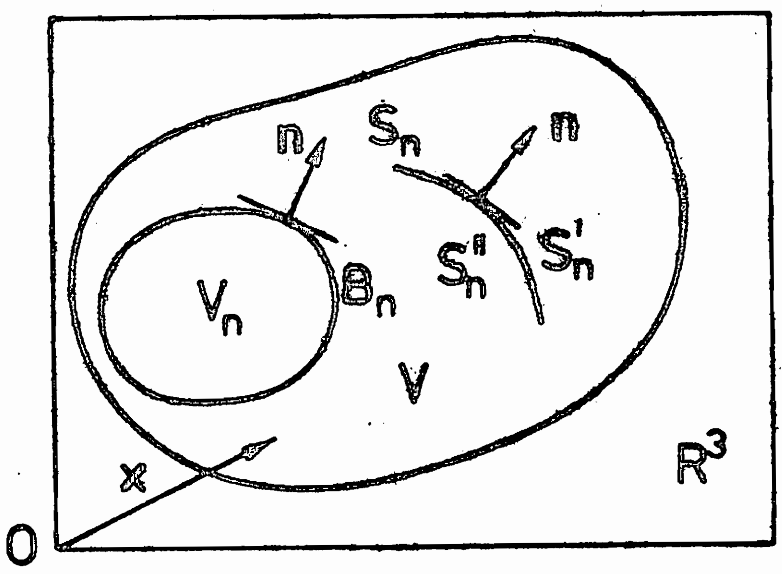

Fig. 2. Regular subregion and regular section of the region V

The compatibility equation in the generalized sense is equivalent to the requirement of compatible displacement of arbitrarily chosen, sufficiently regular two-dimensional section S_n of the region V.

$$\int_{S'_n} u_i \, d\underset{\sim}{x} = \int_{S''_n} u_i \, d\underset{\sim}{x} \quad \text{for every} \quad S_n \subset V \tag{2.4}$$

where S'_n and S''_n denote the surfaces resulting from the section of the region V.

3. CONSTITUTIVE RELATIONS

3.1 Basic concepts

We shall consider behaviour of an element of material assuming the uniform strain and stress distributions represented by tensors $\tilde{\varepsilon}$ and $\tilde{\sigma}$, respectively. Let at the initial moment the element be at the natural, undeformed state (o). Let us apply an external loading $\tilde{\sigma}$ to the element and let us consider its actual state (a) at the moment t_o. The deformation of element is represented by total strain tensor $\tilde{\varepsilon}$. Now, if we unload the element then it relaxes either immediately or asymptotically in time. The tensor $\tilde{\varepsilon}^p$ representing the deformation in the relaxed state (r) is called the permanent or plastic strain. In this work we shall consider materials which give immediate response to unloading. Thus we shall call tensor $\tilde{\varepsilon}^e = \tilde{\varepsilon} - \tilde{\varepsilon}^p$ the elastic strain.

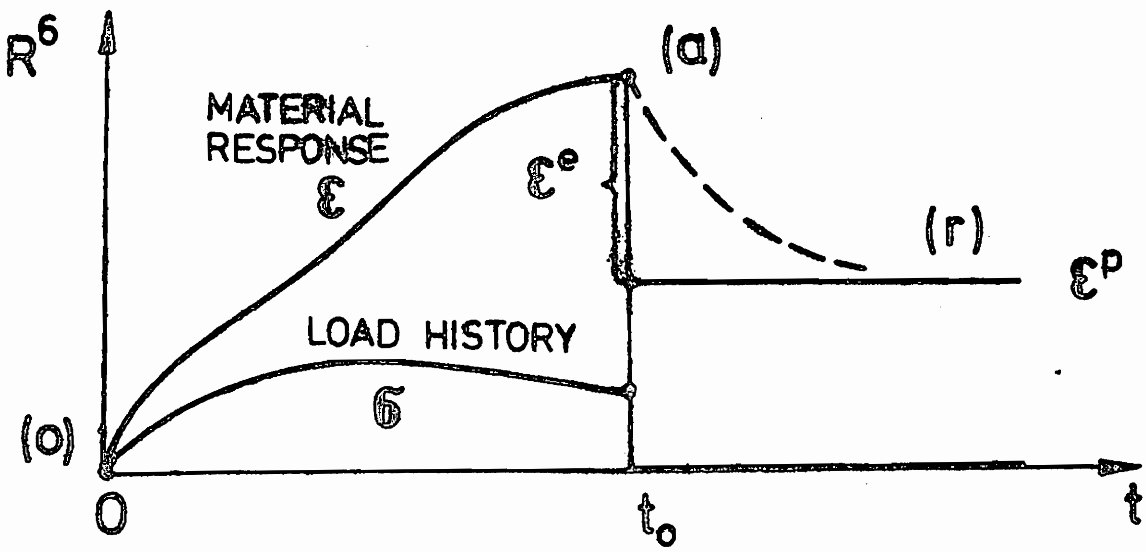

Fig. 3. Elastic and plastic strain components

To describe behaviour of the elastic-viscoplastic body we shall use the concept of internal parameters. We shall assume that the material properties can be fully described by the free energy Ψ and the plastic potential φ. Free energy function is used to describe the elastic behaviour of the material, while the plastic potential characterizes the visco-plastic behaviour.

3.2 Elastic behaviour

We assume that the free energy Ψ depends on the elastic strain tensor $\underset{\sim}{\varepsilon}^e$ and a p-dimensional vector of elastic internal parameters ω_n, n = 1,2,..., p. The couple $\underset{\sim}{e}^e = [\underset{\sim}{\varepsilon}^e, \underset{\sim}{\omega}]$ will be called the generalized elastic strain. We shall assume that the function $\Psi(\underset{\sim}{e}^e)$ is strictly convex, differentiable and that it attains minimum at the origin of (6 + p)-dimensional space $\underset{\sim}{T} = R^{6+p}$. The scalar product in this space is defined by

$$\underset{\sim}{e} \cdot \underset{\sim}{e}^* = \varepsilon_{ij}\varepsilon_{ij}^* + \omega_n\omega_n^* \qquad \underset{\sim}{e}, \underset{\sim}{e}^* \in \underset{\sim}{T} \tag{3.1}$$

The generalized stress tensor $\underset{\sim}{s} = [\underset{\sim}{\sigma}, \underset{\sim}{\pi}]$ corresponding to the generalized elastic strain $\underset{\sim}{e}^e$ is defined as the gradient of Ψ at $\underset{\sim}{e}^e$.

$$\boxed{\underset{\sim}{s} = \frac{\partial \Psi}{\partial \underset{\sim}{e}^e}} \tag{3.2}$$

The above stress-elastic strain relation describes the elastic behaviour of material. The first component of $\underset{\sim}{s}$ is the symmetric stress tensor σ_{ij} and the second will be called the vector of internal force π_n, n = 1,2,...,p.

Making use of the convexity and differentiability of free energy Ψ we can construct the complementary (dual) energy Ψ^* in the generalized stress tensor space $\underset{\sim}{T}$. The complementary energy is obtained by

$$\Psi^*(\underset{\sim}{s}) = \max_{t \in T} [\underset{\sim}{s} \cdot \underset{\sim}{t} - \Psi(\underset{\sim}{t})] \tag{3.3}$$

Now we can express the generalized elastic strain in terms of generalized stress

$$\boxed{\underset{\sim}{e}^e = \frac{\partial \Psi^*}{\partial \underset{\sim}{s}}} \tag{3.4}$$

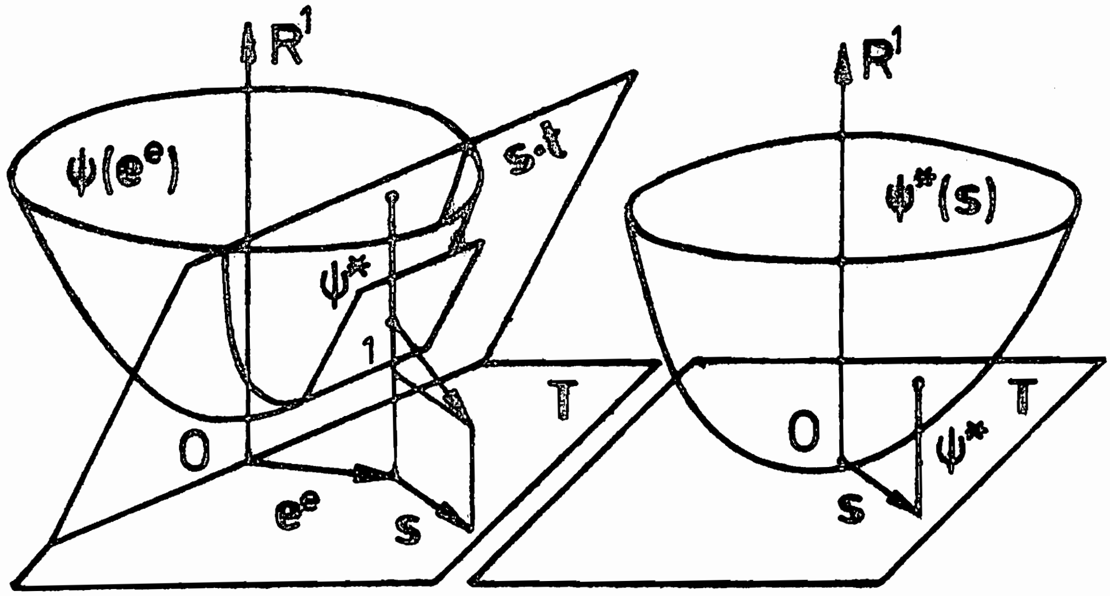

Fig. 4.. Construction of the complementary free energy

3.3 Viscoplastic behaviour

We assume that the plastic potential φ is a lower-semicontinuous, convex function, defined in the generalized stress tensor space $\underset{\sim}{T}$, which attains its minimum at the origin $\underset{\sim}{s} = 0$.

The viscoplastic flow is characterized by the rate of plastic strain tensor $\dot{\underset{\sim}{\varepsilon}}^p$ and the rate of plastic internal parameter vector $\dot{\kappa}_n$, $n = 1,2,\ldots,p$. The couple $\underset{\sim}{e}^p = [\underset{\sim}{\varepsilon}^p, \underset{\sim}{\kappa}]$ will be called the generalized plastic strain. The plastic flow law is based on normal dissipation hypothesis and can be formulated with the relation

$$\boxed{\dot{\underset{\sim}{e}}^p \in \partial\varphi(\underset{\sim}{s})} \qquad (3.5)$$

where $\partial\varphi(\underset{\sim}{s})$ is the subdifferential of the plastic potential φ at $\underset{\sim}{s}$. The subdifferential $\partial\varphi(\underset{\sim}{s})$ is defined as the set of all subgradients of φ at $\underset{\sim}{s}$. The generalized tensor $\dot{\underset{\sim}{e}}^p \in \underset{\sim}{T}$ is called the subgradient of φ at $\underset{\sim}{s}$ if

$$\dot{\underset{\sim}{e}}^{p} \cdot \underset{\sim}{s} - \varphi(\underset{\sim}{s}) = \varphi^{*}(\dot{\underset{\sim}{e}}^{p}) \qquad (3.6)$$

where $\varphi^{*}(\dot{\underset{\sim}{e}}^{p})$ is the polar potential defined by

$$\varphi^{*}(\dot{\underset{\sim}{e}}^{p}) = \sup_{\underset{\sim}{t} \in \underset{\sim}{T}} [\dot{\underset{\sim}{e}}^{p} \cdot \underset{\sim}{t} - \varphi(\underset{\sim}{t})] \qquad (3.7)$$

The product $\dot{\underset{\sim}{e}}^{p} \cdot \underset{\sim}{s}$ represents the rate of energy dissipation during the plastic flow.

It follows from the above considerations that we can express the plastic flow law in the dual form

$$\boxed{\underset{\sim}{s} \in \partial\varphi^{*}(\dot{\underset{\sim}{e}}^{p})} \qquad (3.8)$$

It should be noted that the subdifferential is always defined in every point of the space $\underset{\sim}{T}$. We shall distinguish four particular cases:

(i) subdifferential is identical with the entire space $\underset{\sim}{T}$

(ii) subdifferential is a convex region in the space $\underset{\sim}{T}$

(iii) subdifferential consists of one tensor from $\underset{\sim}{T}$ only

(iv) subdifferential is the empty set.

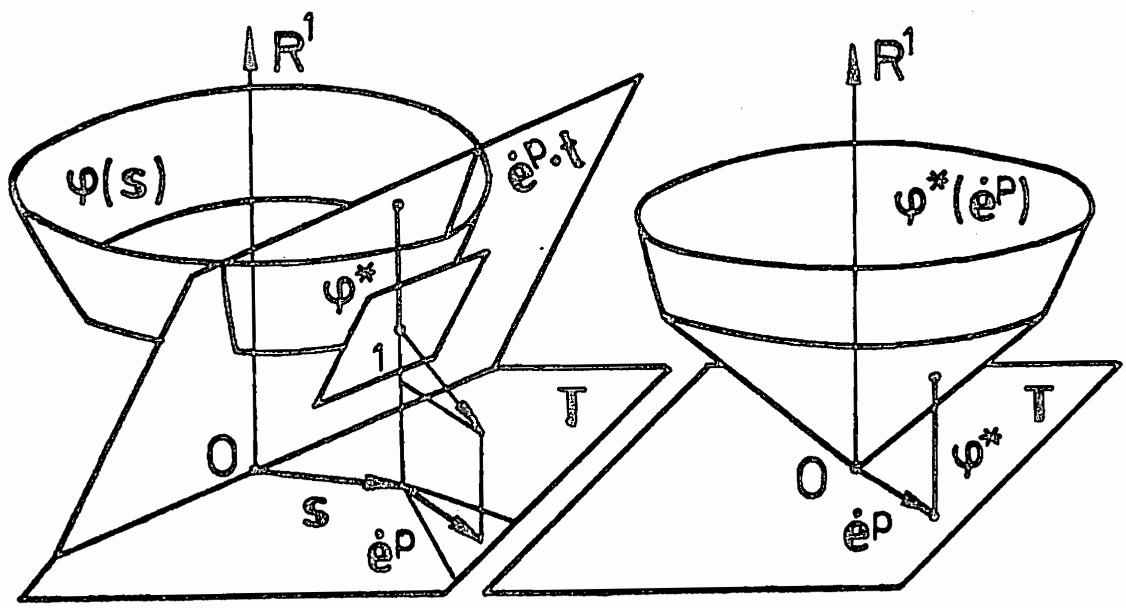

Fig. 5. Construction of the polar potential

3.4 Relation between internal parameters

We postulate that the rate of total work done on the element of the body is expressed by the product $\underset{\sim}{s} \cdot \dot{\underset{\sim}{e}}$ where $\underset{\sim}{e}$ is the generalized total strain tensor

$$\underset{\sim}{e} = \underset{\sim}{e}^{e} + \underset{\sim}{e}^{p} \tag{3.9}$$

On the other hand the rate of total work is equal to $\underset{\sim}{\sigma} \cdot \dot{\underset{\sim}{\varepsilon}}$ as the energy is supplied to the element by the external forces only. It follows from $\underset{\sim}{s} \cdot \dot{\underset{\sim}{e}} = \underset{\sim}{\sigma} \cdot \dot{\underset{\sim}{\varepsilon}}$ that $\dot{\underset{\sim}{\omega}} + \dot{\underset{\sim}{\kappa}} = 0$. Hence, if we assume that $\underset{\sim}{\omega} = \underset{\sim}{\kappa} = 0$ at the initial moment, we obtain

$$\underset{\sim}{\omega} + \underset{\sim}{\kappa} = 0 \tag{3.10}$$

i. e. the elastic and plastic internal parameters are coupled.

3.5 Standard material model

In the present work we shall consider the GENERALIZED STANDARD elastic-viscoplastic MATERIAL model. Such material is characterized by the quadratic form of the complementary free energy

$$\Psi^{*}(\underset{\sim}{s}) = \frac{1}{2}\, \underset{\sim}{s} \cdot \underset{\sim}{G}\, \underset{\sim}{s} \tag{3.11}$$

where $\underset{\sim}{G}$ is the generalized matrix of elastic coefficients which consists of matrix L_{ijkl} of elastic coefficients and the matrix Z_{mn}, m, n = 1,2,...,p of internal elastic coefficients. The product $\underset{\sim}{G}\,\underset{\sim}{s}$ is defined by

$$\underset{\sim}{G}\,\underset{\sim}{s} = [L_{ijkl}\sigma_{kl},\ Z_{mn}\pi_{n}] \quad \text{if} \quad \underset{\sim}{s} = [\underset{\sim}{\sigma},\underset{\sim}{\pi}] \tag{3.12}$$

Hence the stress-strain relation assumes the form of generalized Hooke's law

$$\boxed{\underset{\sim}{e}^{e} = \underset{\sim}{G}\,\underset{\sim}{s}} \quad \text{or} \quad \begin{aligned} \underset{\sim}{\varepsilon}^{e} &= \underset{\sim}{L}\,\underset{\sim}{\sigma} \\ \underset{\sim}{\omega} &= \underset{\sim}{Z}\,\underset{\sim}{\pi} \end{aligned} \tag{3.13}$$

We shall distinguish the particular case of GENERALIZED STANDARD elastic-plastic MATERIAL where the plastic potential is prescribed in $\underset{\sim}{T}$ in the form of indicator function, i. e. it assumes only two values: 0 and + ∞. It follows from the properties of the plastic potential that the region $\underset{\sim}{E}_t$, where $\varphi(\underset{\sim}{s}) = 0$, is convex and contains the origin of $\underset{\sim}{T}$. The region $\underset{\sim}{E}_t$ is called the elastic region in the stress space $\underset{\sim}{T}$.

Now the plastic flow law may be formulated in terms of the elastic region. Namely, the tensor $\underset{\sim}{s}$ of generalized stress and the tensor $\dot{\underset{\sim}{e}}^p$ of plastic strain rate satisfy the plastic flow law if

$$\dot{\underset{\sim}{e}}^p \cdot \underset{\sim}{s} = \varphi^*(\dot{\underset{\sim}{e}}^p) \tag{3.14}$$

Here $\varphi^*(\dot{\underset{\sim}{e}}^p)$ represents the rate of energy dissipation during the plastic flow. The plastic flow law can be also presented in the form of inequality

$$\dot{\underset{\sim}{e}}^p \cdot \underset{\sim}{s} \geq \dot{\underset{\sim}{e}}^p \cdot \underset{\sim}{t} \quad \text{for every } \underset{\sim}{t} \in \bar{\underset{\sim}{E}}_t \tag{3.15}$$

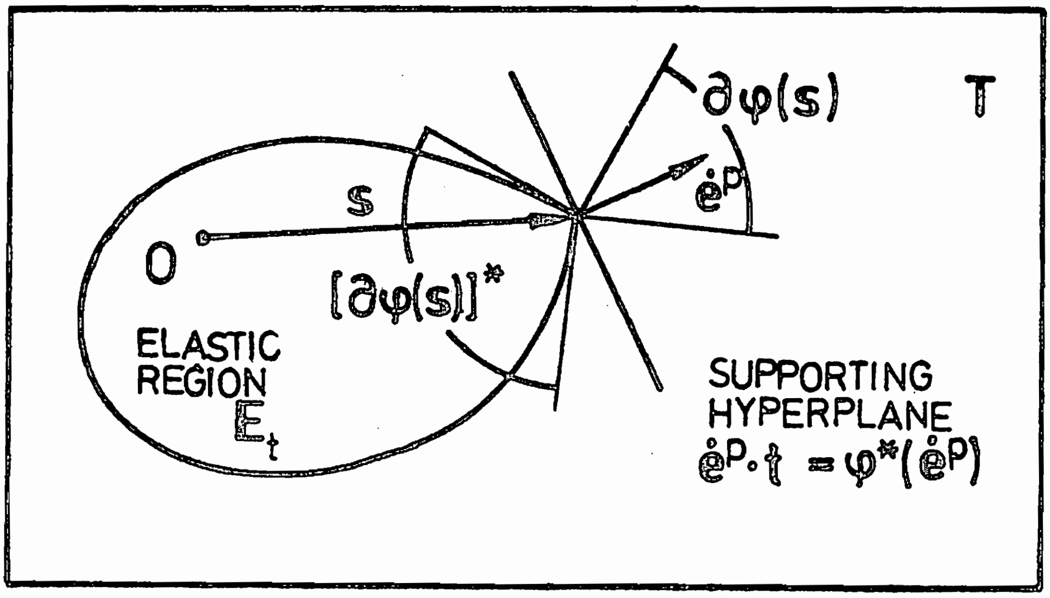

Fig. 6. Plastic flow law for the elastic-plastic model

The subdifferential $\partial\varphi(\underset{\sim}{s})$ can be now interpreted as the cone of all generalized tensors $\dot{\underset{\sim}{e}}^p$ which determine a hyperplane $\dot{\underset{\sim}{e}}^p \cdot \underset{\sim}{t} = \varphi^*(\dot{\underset{\sim}{e}}^p)$ supporting the region $\underset{\sim}{E}_t$ at the point $\underset{\sim}{s}$.

It can be shown that during the plastic deformation

(i) plastically admissible rate of generalized stress tensor $\dot{\underset{\sim}{s}}$ belongs to the polar cone $[\partial\varphi(\underset{\sim}{s})]^*$ defined by

$$[\partial\varphi(\underset{\sim}{s})]^* = [\underset{\sim}{t}^* : \underset{\sim}{t}^* \cdot \underset{\sim}{t} \leq 0 \quad \text{for every} \quad \underset{\sim}{t} \in \partial\varphi(\underset{\sim}{s})] \qquad (3.16)$$

(ii) the generalized plastic strain rate is always orthogonal to the generalized stress rate.

$$\dot{\underset{\sim}{e}}^p \cdot \dot{\underset{\sim}{s}} = 0 \qquad (3.17)$$

In the particular case, when the free energy Ψ and the plastic potential φ do not depend on internal parameters we arrive at the STANDARD elastic-perfectly plastic MATERIAL. In this case all generalized tensors are replaced by the corresponding six-dimensional symmetric tensors defined in the space $T = R^6$, and the matrix $\underset{\sim}{G}$ is replaced by $\underset{\sim}{L}$.

3.6 Regular standard material

In our considerations of standard material models we shall distinguish the regular material which is characterized by particular properties of the plastic potential φ. Namely, the material will be called regular if for every generalized stress tensor $\underset{\sim}{s} \in \bar{\underset{\sim}{E}}_t$ and for every $\dot{\underset{\sim}{\kappa}}$ satisfying the relation

$$[\dot{\underset{\sim}{\varepsilon}}^p, \dot{\underset{\sim}{\kappa}}] \in \partial\varphi(\underset{\sim}{s}) \qquad (3.18)$$

the intersection $1(\dot{\underset{\sim}{\kappa}}) \cap \partial\varphi(\underset{\sim}{s})$ consists of one generalized strain rate tensor only. Here $1(\underset{\sim}{\vartheta})$, defined for every $\underset{\sim}{\vartheta} \in R^p$, is the set of all generalized tensors $\underset{\sim}{t} = [\underset{\sim}{\tau}, \underset{\sim}{\vartheta}]$ which have the second component equal to $\underset{\sim}{\vartheta}$

$$l(\underset{\sim}{\vartheta}) = [\underset{\sim}{t} : \underset{\sim}{t} = [\underset{\sim}{\tau},\underset{\sim}{\vartheta}] \, , \underset{\sim}{\tau} \in R^6] \qquad (3.19)$$

It follows from the above definition that in the case of regular material the plastic strain rate tensor $\dot{\underset{\sim}{\varepsilon}}^p$ is uniquely determined by the generalized stress tensor $\underset{\sim}{s}$ and by the rate of plastic (or elastic) internal parameter vector $\dot{\underset{\sim}{\kappa}}$ (or $\dot{\underset{\sim}{\omega}}$). We shall use this property to discuss the uniqueness of strain.

It should be mentioned that the standard elastic-perfectly plastic material is not regular as the intersection $l(\underset{\sim}{O}) \cap \partial\varphi(\underset{\sim}{s})$ contains entire cone $\partial\varphi(\underset{\sim}{s})$

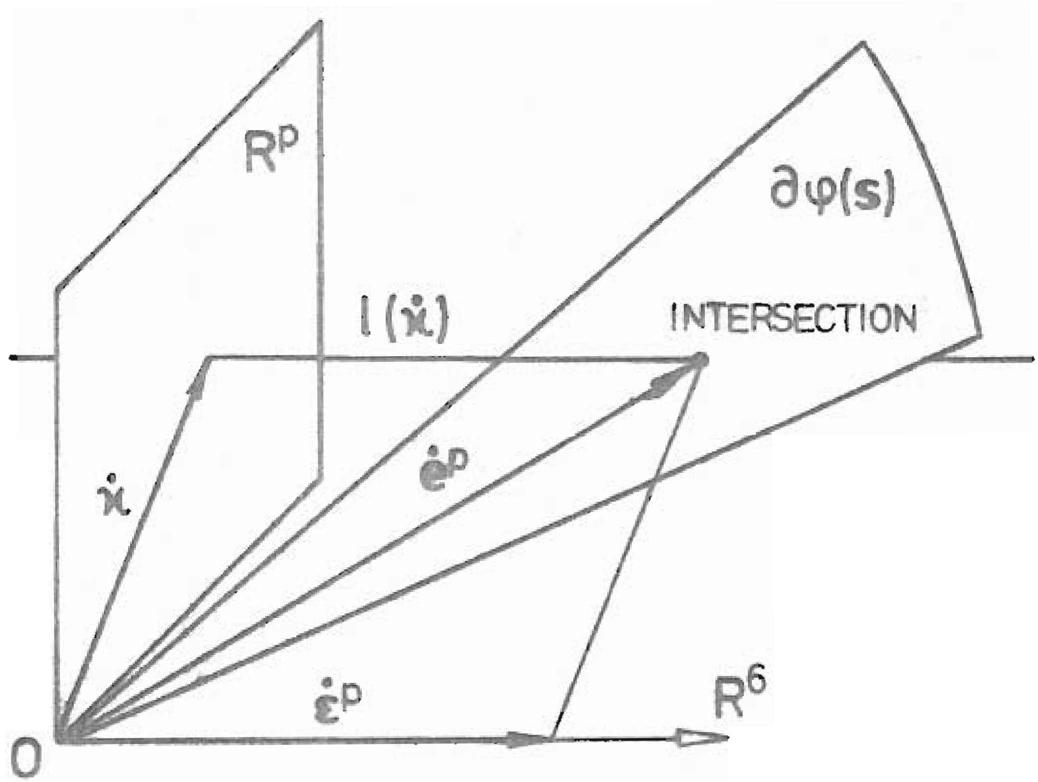

Fig. 7. Cone of plastic strain rates for the regular material

3.7 Examples of standard materials

The one-dimensional model of generalized standard elastic-viscoplastic material can be constructed of springs, slides representing dry friction and dashpots representing viscous damping. Simple one-parameter one-dimensional material models are presented below

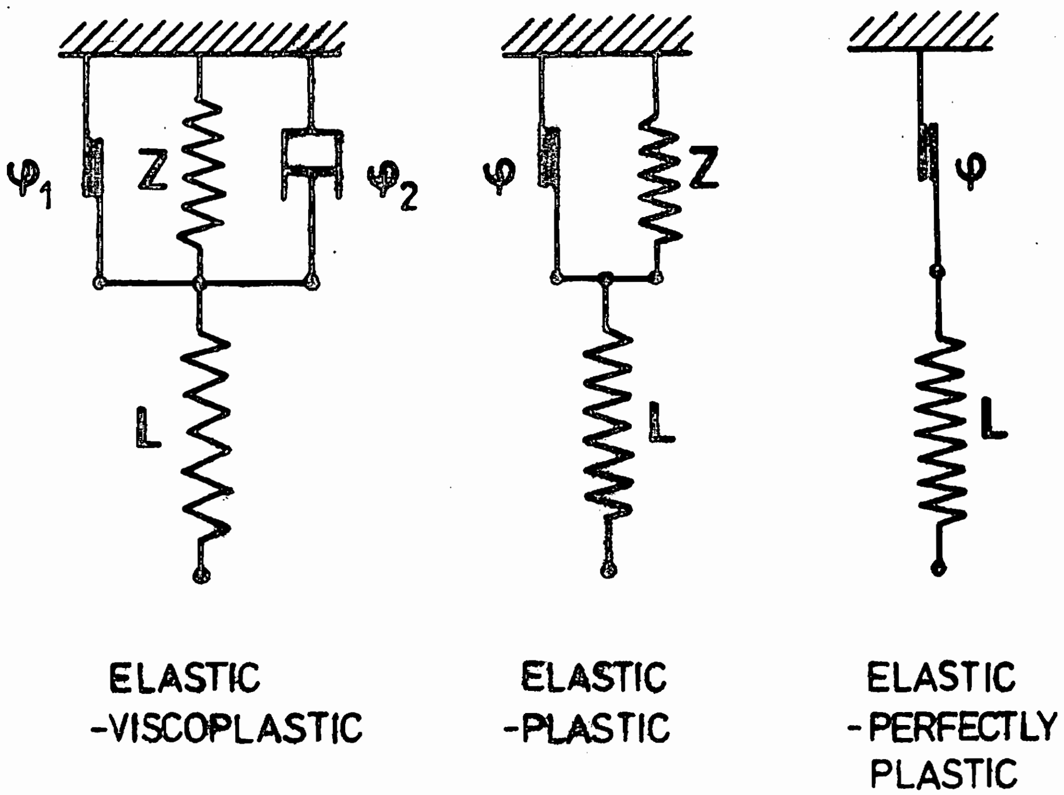

Fig. 8. One-parameter one-dimensional standard material models

It should be noted that the slide is characterized by the plastic potential φ_1, which has the form of indicator function while the dashpot is characterized by the viscous potential φ_2 in the form of differentiable function. The behaviour of parallely connected slide and dashpot is represented by the viscoplastic potential $\varphi = \varphi_1 + \varphi_2$.

We can give the mechanical interpretation of internal parameters and internal forces considering displacements and forces in the elements of the one-dimensional structure.

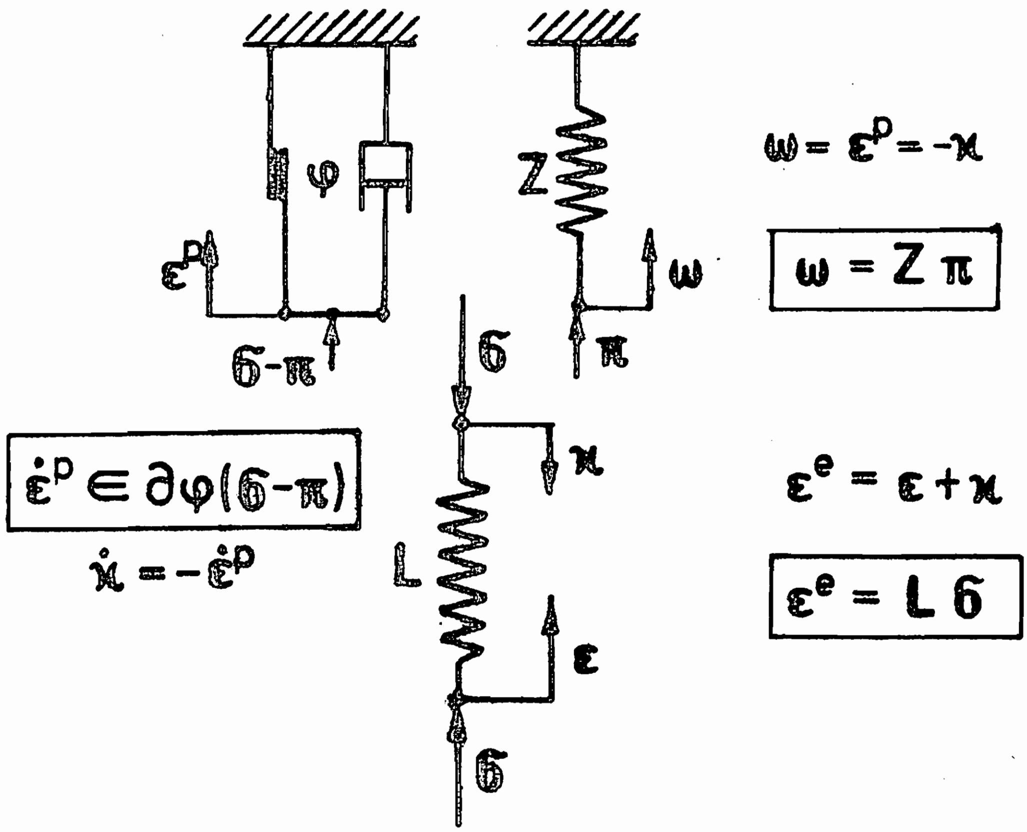

Fig. 9. Internal parameters and internal forces for one-dimensional material model.

In the case of elastic-plastic material the elastic region takes the form of diagonal strip in the generalized stress space

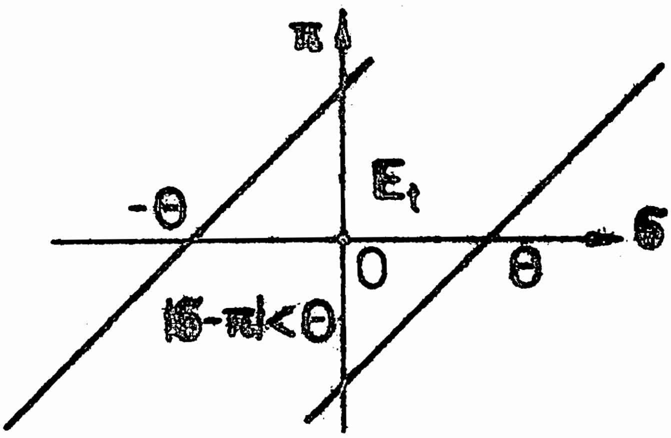

Fig. 10. Elastic region for one-dimensional elastic-plastic material

More complex, multiple parameter one-dimensional standard models can be constructed of dashpots, slides and springs connected parallely or in series.

The generalized standard material model includes also three-dimensional elastic-plastic models used for pratical calculations. Namely, using the internal parameter concept we can describe the Mises work-hardening material

$$\underset{\sim}{E}_t = [\underset{\sim}{s} = [\underset{\sim}{\sigma},\underset{\sim}{\pi}] : |\underset{\sim}{\sigma}' - \underset{\sim}{\beta}| < \pi_1 , \beta_{11} = \pi_2 , \beta_{21} = \pi_3, \beta_{31} = \pi_4 , \beta_{22} = \pi_5 , \beta_{23} = \pi_6 , \beta_{33} = \pi_7] \tag{3.20}$$

where $\underset{\sim}{\sigma}'$ is the deviator of the tensor $\underset{\sim}{\sigma}$

$$\sigma'_{ij} = \sigma_{ij} - (\sigma_{kk}/3)\delta_{ij} \qquad \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{3.21}$$

the internal force π_1 represents the yield limit and the symmetric tensor $\underset{\sim}{\beta}$ denotes the displacement of the center of elastic region during the plastic flow. The internal parameter $\underset{\sim}{\omega}$ is defined by

$$\omega_1 = \int_0^t |\dot{\varepsilon}^p| dt , \omega_2 = \varepsilon^p_{11} , \omega_3 = \varepsilon^p_{21} , \omega_4 = \varepsilon^p_{31} , \omega_5 = \varepsilon^p_{22} , \omega_6 = \varepsilon^p_{32} , \omega_7 = \varepsilon^p_{33} \tag{3.22}$$

The relation between the internal parameter and internal force is determined by a positive definite matrix $\underset{\sim}{Z}$ such that

$$Z_{11} = k \quad Z_{1i} = Z_{i1} = 0 \quad i = 2,\ldots,7. \tag{3.23}$$

Similarly we can describe the Tresca work hardening material in terms of generalized standard material concepts

$$\underset{\sim}{E}_t = [\underset{\sim}{s} = [\underset{\sim}{\sigma},\underset{\sim}{\pi}]: \min_{J \neq I} |\sigma'_{(I)} - \sigma'_{(J)} - \beta_I| < \pi_1 ,\ I = 1,2,3 ,$$
$$\beta_1 = \pi_2 ,\ \beta_2 = \pi_3 ,\ \beta_3 = \pi_4] \tag{3.24}$$

where $\sigma'_{(I)}$ denotes the I-th principal value of the stress deviator $\underset{\sim}{\sigma}'$. Now the internal parameter $\underset{\sim}{\omega}$ is defined by

$$\omega_1 = \int_0^t |\dot{\underset{\sim}{\varepsilon}}^p| dt ,\ \omega_2 = \varepsilon^p_{(1)} ,\ \omega_3 = \varepsilon^p_{(2)} ,\ \omega_4 = \varepsilon^p_{(3)} \tag{3.25}$$

and a positive-definite matrix $\underset{\sim}{Z}$ satisfies the relations

$$Z_{11} = k \quad Z_{1i} = Z_{i1} = 0 \quad i = 2,3,4. \tag{3.26}$$

4. BOUNDARY VALUE PROBLEM

Primarily we formulate the elastic-viscoplastic boundary problem in four-dimensional space-time region $\underset{\sim}{V}$. We shall assume, for simplicity, that all fields defined in $\underset{\sim}{V}$ are smooth (i. e. they have all derivatives with respect to $\underset{\sim}{x}$ and t) and that they vanish at t = O.

4.1 Smooth kinematically and statically admissible fields

We introduce the kinematically admissible stress field $\underset{\sim}{\mu}$ which satisfies the compatibility equation

$$\gamma_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}) \quad \text{in } \underset{\sim}{V} \tag{4.1}$$

$$\gamma_{ij} = L_{ijkl}\mu_{kl} \quad \text{in } \underset{\sim}{V} \tag{4.2}$$

(i. e. there exists the displacement field $\underset{\sim}{v}$ defined in $\underset{\sim}{V}$ such that $L_{ijkl}\mu_{kl} = \frac{1}{2}(v_{i,j} + v_{j,i})$ in $\underset{\sim}{V}$) and vanishes on the boundary $\underset{\sim}{B}_k$

$$v_i = 0 \quad \text{on } \underset{\sim}{B}_k \tag{4.3}$$

We shall also use the concept of statically admissible stress field $\underset{\sim}{\rho}$ which satisfies the equilibrium equation

$$\rho_{ij,j} = 0 \quad \text{in } \underset{\sim}{V} \tag{4.4}$$

and vanishes on the boundary $\underset{\sim}{B}_s$

$$\rho_{ij}n_j = 0 \quad \text{on } \underset{\sim}{B}_s \tag{4.5}$$

It can be shown that for every symmetric tensor field $\underset{\sim}{\mu}$ satisfying (4.1), (4.2) and for every differentiable symmetric tensor field $\underset{\sim}{\rho}$ we have

$$\int_V \rho_{ij}L_{ijkl}\mu_{kl}\, d\underset{\sim}{x} = \int_B \rho_{ij}n_j v_i\, d\underset{\sim}{x} - \int_V \rho_{ij,j}v_i\, d\underset{\sim}{x} \tag{4.6}$$

Hence if $\underset{\sim}{\rho}$ is statically admissible and $\underset{\sim}{\mu}$ is kinematically admissible then the scalar product

$$(\underset{\sim}{\rho},\underset{\sim}{\mu})_L \overset{df}{=} \int_V \rho_{ij} L_{ijkl} \mu_{kl} \, d\underset{\sim}{x} = 0 \tag{4.7}$$

i. e. every kinematically admissible field $\underset{\sim}{\mu}$ is orthogonal, with respect to the L-scalar product to every statically admissible field $\underset{\sim}{\rho}$.

In further considerations we shall use the concepts of generalized statically admissible field $\underset{\sim}{r}$ and generalized kinematically admissible field $\underset{\sim}{m}$. The field $\underset{\sim}{r} = [\underset{\sim}{\rho},\underset{\sim}{\vartheta}]$ is called statically admissible if $\underset{\sim}{\rho}$ is statically admissible. The field $\underset{\sim}{m} = [\underset{\sim}{\mu},\underset{\sim}{\nu}]$ is called kinematically admissible if $\underset{\sim}{\mu}$ is kinematically admissible and $\underset{\sim}{\nu} = 0$ in $\underset{\sim}{V}$. The G-scalar product of two generalized fields $\underset{\sim}{t} = [\underset{\sim}{\tau},\underset{\sim}{\vartheta}]$ and $\underset{\sim}{t}^* = [\underset{\sim}{\tau}^*,\underset{\sim}{\vartheta}^*]$ is defined by

$$(\underset{\sim}{t},\underset{\sim}{t}^*)_G \overset{df}{=} \int_V \underset{\sim}{t} \cdot \underset{\sim}{G}\underset{\sim}{t}^* \, d\underset{\sim}{x} = \int_V (\underset{\sim}{\tau} \cdot \underset{\sim}{L}\underset{\sim}{\tau}^* + \underset{\sim}{\vartheta} \cdot \underset{\sim}{Z}\underset{\sim}{\vartheta}^*) \, d\underset{\sim}{x} \tag{4.8}$$

It follows from the definitions that every generalized statically admissible field $\underset{\sim}{r}$ is G-orthogonal to every generalized kinematically admissible field $\underset{\sim}{m}$.

$$(\underset{\sim}{r},\underset{\sim}{m})_G = 0 \tag{4.9}$$

The above scalar products are defined for fixed time t_o and they will be used to describe the rate boundary value problem.

We shall also introduce the G-scalar product constructed with space-time integration over region $\underset{\sim}{V}$ appropriate to the initial-boundary value problem

$$<\underset{\sim}{t},\underset{\sim}{t}^*>_G \overset{df}{=} \int_{\underset{\sim}{V}} \underset{\sim}{t} \cdot \underset{\sim}{G}\underset{\sim}{t}^* \, d\underset{\sim}{x} \, e^{-t} \, dt \tag{4.10}$$

Let us observe that the integration over time does not change the properties of statically and kinematically admissible fields, i. e. we have

$$\langle \underset{\sim}{r}, \underset{\sim}{m} \rangle_G = 0 \tag{4.11}$$

for every statically admissible $\underset{\sim}{r}$ and kinematically admissible $\underset{\sim}{m}$.

The function e^{-t} appearing in the definition of the scalar product provides us with a simple relation between the field $\underset{\sim}{t}$ and its time derivative $\dot{\underset{\sim}{t}}$.

$$2\langle \underset{\sim}{t}, \dot{\underset{\sim}{t}} \rangle_G = \langle \underset{\sim}{t}, \underset{\sim}{t} \rangle_G \tag{4.12}$$

provided that the initial value of the field $\underset{\sim}{t}$ vanishes.

4.2 Perfectly elastic solution

In the sequel we shall assume that the solution $\underset{\sim}{\sigma}^o$ of the perfectly elastic boundary value problem is known, i. e. we can find the stress field $\underset{\sim}{\sigma}^o$ which satisfies

Equilibrium equation	$\sigma^o_{ij,j} = b_i$	in $\underset{\sim}{V}$	
Boundary force condition	$\sigma^o_{ij}n_j = f_i$	on $\underset{\sim}{B}_s$	
Compatibility equation	$\varepsilon^o_{ij} = \frac{1}{2}(u^o_{i,j} + u^o_{j,i})$	in $\underset{\sim}{V}$	(4.13)
Boundary displacement conditions	$u^o_i = p_i$	on $\underset{\sim}{B}_k$	
Hooke's law	$\varepsilon^o_{ij} = L_{ijkl}\sigma^o_{kl}$	in $\underset{\sim}{V}$	

The perfectly elastic solution $\underset{\sim}{\sigma}^o$ represents all external loading applied to the body i. e. boundary force $\underset{\sim}{f}$ boundary displacement $\underset{\sim}{p}$ and body force $\underset{\sim}{b}$.

Consequently we introduce the concept of generalized perfectly elastic solution $\underset{\sim}{s}^o = [\underset{\sim}{\sigma}^o, \underset{\sim}{0}]$ which will be used to construct the actual generalized stress field $\underset{\sim}{s}$.

4.3 Initial-boundary value problem

Now the initial-boundary value problem can be formulated as follows:

> Find the generalized stress field $\underset{\sim}{s} = [\underset{\sim}{\sigma},\underset{\sim}{\pi}]$ and the generalized total strain field $\underset{\sim}{e} = [\underset{\sim}{\varepsilon},\underset{\sim}{0}]$, defined in the space time region $\underset{\sim}{V}$ and vanishing at the initial moment, such that
>
> (i) the generalized residual stress field $\underset{\sim}{r} = \underset{\sim}{s}^{o} - \underset{\sim}{s}$ is statically admissible
>
> (ii) the generalized stress field $\underset{\sim}{m} = \underset{\sim}{G}^{-1}\underset{\sim}{e} - \underset{\sim}{s}^{o}$ is kinematically admissible
>
> (iii) the generalized plastic strain rate field $\dot{\underset{\sim}{e}}^{p} = \underset{\sim}{G}(\dot{\underset{\sim}{r}} + \dot{\underset{\sim}{m}})$ and the generalized stress field $\underset{\sim}{s}$ satisfy the plastic flow law.

It should be noted that the fields $\underset{\sim}{s}$ and $\underset{\sim}{e}$ at a given moment t_o depend not only on loading represented by $\underset{\sim}{s}^{o}$ at t_o, but also on the history of viscoplastic deformation in the time interval $[0,t_o]$. For that reason the initial-boundary value problem is considered in four-dimensional space-time region $\underset{\sim}{V}$. The solution of the initial-boundary value problem represents the entire history of generalized stress and strain in the region V.

4.4 Rate boundary value problem

The majority of works on the variational approach to the elastic-viscoplastic problem, which have appeared till now concern the rate boundary value problem. The solution of the rate boundary value problem gives us only the rates of generalized stress and strains at the moment t_o provided that all fields at t_o are known and are not subjected to any variation. Consequently we consider both the field rates and the field values at time t_o as tensor functions defined in three-dimensional region V. The other necessary assumption is that the time derivatives of fields exist in every moment $t_o \geq 0$.

The rate boundary value problem can be formulated as follows:

> The generalized stress field $\underset{\sim}{s} = [\underset{\sim}{\sigma}, \underset{\sim}{\pi}]$ and the generalized total strain field $\underset{\sim}{e} = [\underset{\sim}{\varepsilon}, \underset{\sim}{\rho}]$ defined in the region V at given moment t_o are known. Find the rate of generalized stress field $\dot{\underset{\sim}{s}} = [\dot{\underset{\sim}{\sigma}}, \dot{\underset{\sim}{\pi}}]$ and the rate of generalized total strain field $\dot{\underset{\sim}{e}} = [\dot{\underset{\sim}{\varepsilon}}, \dot{\underset{\sim}{\rho}}]$ defined in the region V such that
>
> (i) the rate of generalized residual stress field $\dot{\underset{\sim}{r}} = \dot{\underset{\sim}{s}}^o - \dot{\underset{\sim}{s}}$ is statically admissible
>
> (ii) the rate of generalized stress field $\dot{\underset{\sim}{m}} = G^{-1}\dot{\underset{\sim}{e}} - \dot{\underset{\sim}{s}}^o$ is kinematically admissible
>
> (iii) the generalized plastic strain rate field $\dot{\underset{\sim}{e}}^p = \underset{\sim}{G}(\dot{\underset{\sim}{r}} + \dot{\underset{\sim}{m}})$ and the prescribed generalized stress field $\underset{\sim}{s}$ satisfy the plastic flow law.

The solution of the rate boundary value problem can be directly used to construct a numerical solution with the step-by-step (in time) method. In this case the history of plastic deformation is represented by the generalized stress and strains at the considered moment t_o.

5. SOLUTION OF THE INITIAL-BOUNDARY VALUE PROBLEM

5.1 Construction of the space of admissible fields

To complete the formulation of the initial-boundary value problem given in the previous section we shall define the space of generalized stress field where the solution of the problem is to be found. Namely, we shall construct the Hilbert space $\underset{\sim}{H}$ of all generalized tensor fields $\underset{\sim}{t}$ which have finite G-norm

$$|||\underset{\sim}{t}|||_G^2 \overset{df}{=} \int_{\underset{\sim}{V}} \underset{\sim}{t} \cdot \underset{\sim}{G}\,\underset{\sim}{t}\, d\underset{\sim}{x}\, e^{-t} dt < +\infty \qquad (5.1)$$

We shall call $\underset{\sim}{H}$ the <u>space of admissible fields</u> and we shall seek our solution in this space.

To construct the Hilbert space $\underset{\sim}{H}$ it is convenient to start with the space $\underset{\sim}{C}^{\infty}(\underset{\sim}{V})$ of all <u>smooth</u> (having all derivatives with respect to Cartesian coordinates and time) generalized tensor fields <u>bounded</u> in the space-time region $\underset{\sim}{V}$ and vanishing at $t = 0$. The norm in the space $\underset{\sim}{C}^{\infty}(\underset{\sim}{V})$ is given by

$$\|\underset{\sim}{t}\|_C^2 \overset{df}{=} \sup_{(\underset{\sim}{x},t)\in \underset{\sim}{V}} \underset{\sim}{t} \cdot \underset{\sim}{t} < +\infty \qquad (5.2)$$

The space $\underset{\sim}{C}^{\infty}(\underset{\sim}{V})$ provided with the G-norm is called the <u>unitary space</u>. It follows from the fact that the matrix $\underset{\sim}{G}$ is positive definite that

$$|||\underset{\sim}{t}|||_G \leq c\, |||\underset{\sim}{t}||| \leq c\, |V|\, \|\underset{\sim}{t}\|_C \qquad c \succ 0 \qquad (5.3)$$

where

$$|||\underset{\sim}{t}|||^2 \overset{df}{=} \langle \underset{\sim}{t},\underset{\sim}{t}\rangle \overset{df}{=} \int_{\underset{\sim}{V}} \underset{\sim}{t} \cdot \underset{\sim}{t}\, d\underset{\sim}{x}\, e^{-t}\, dt \qquad (5.4)$$

i. e. the G-norm exists for every field from the unitary space.

The sequence $\{\underset{\sim}{t}_{(n)}\}$ of fields from unitary space is called the Cauchy sequence if for every real $c > 0$ we can find such integer k that

$$|||\, \underset{\sim}{t}_{(n)} - \underset{\sim}{t}_{(k)} \,|||_G < c \qquad \text{for every } n > k \tag{5.5}$$

The limit $\underset{\sim}{t} = \lim_{n\to\infty} \underset{\sim}{t}_{(n)}$ of Cauchy sequence in general does not belong to the unitary space. All limits of the Cauchy sequences can be divided into classes of equivalence. Two limits $\underset{\sim}{t}$ and $\underset{\sim}{t}^*$ belong to the same class of equivalence if the corresponding Cauchy sequences satisfy the relation

$$\lim_{n\to\infty} |||\, \underset{\sim}{t}_{(n)} - \underset{\sim}{t}^*_{(n)} \,|||_G = 0 \tag{5.6}$$

The Hilbert space $\underset{\sim}{H}$ is defined as the set of all classes of equivalence, which can be constructed of fields from unitary space with the G-norm. It should be noted that the space $\underset{\sim}{H}$ provided with the G-norm is complete as it was constructed by completion of the unitary space with respect to that norm. The class of equivalence is fully determined by one field belonging to that class called the representative of the class.

The above construction of the Hilbert space $\underset{\sim}{H}$ provided with the G-norm is not unique. We could alternatively start from other unitary space which assures the existence of the G-norm. For example we can consider the unitary space of all finitely-valued generalized fields defined in the space-time region $\underset{\sim}{V}$.

To define the finite valued field we introduce the family M of all subregions $\underset{\sim}{V}_n$ of the space-time region $\underset{\sim}{V}$ such that

(i) $\underset{\sim}{V} \in M$

(ii) if $\underset{\sim}{V}_n \in M$ then $\underset{\sim}{V} - \underset{\sim}{V}_n \in M$

(iii) if $\underset{\sim}{V}_n \in M$ for $n = 1,2,\ldots$ then the union $\bigcup_{n=1}^{\infty} \underset{\sim}{V}_n \in M$

We shall construct the subregion $\underset{\sim}{V}_n$ of disjoint four-dimensional space-time parallelepipeds $\underset{\sim}{\Lambda}_1$

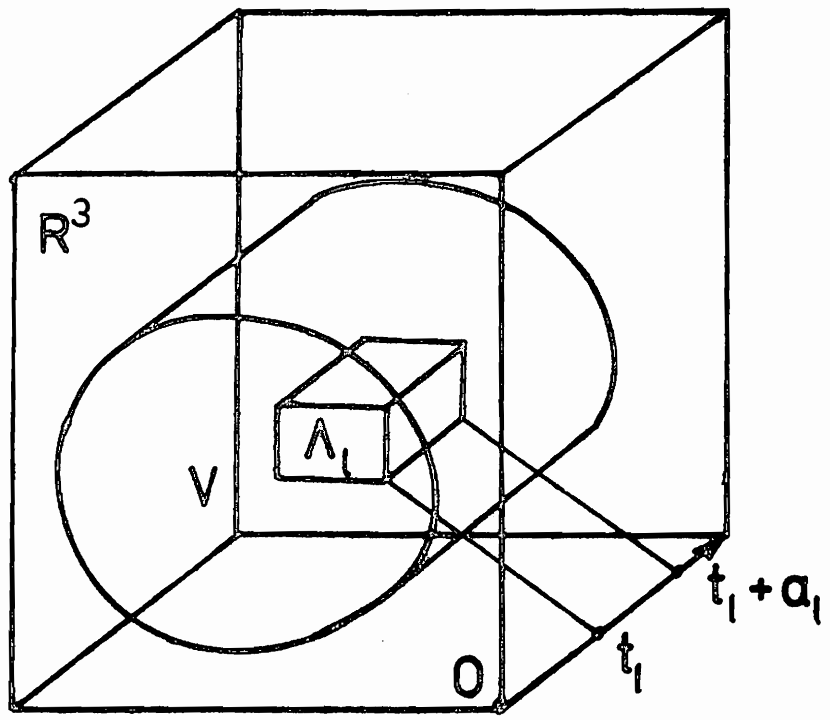

Fig. 11. Four-dimensional space-time parallelepiped

The measure $m(\underset{\sim}{\Lambda}_1)$ of such parallelepiped is defined by

$$m(\underset{\sim}{\Lambda}_1) = |\Lambda_1| \, [\exp(-t_1) - \exp(-t_1 - a_1)] \tag{5.7}$$

The measure $m(\underset{\sim}{V}_n)$ of the subregion $\underset{\sim}{V}_n$ composed of disjoint parallelepipeds $\underset{\sim}{\Lambda}_l$, $l = 1,2,\ldots$ is defined by

$$m(\underset{\sim}{V}_n) = \sum_l m(\underset{\sim}{\Lambda}_l) \tag{5.8}$$

Now we can assign a tensor $\underset{\sim}{t}_n$ from $\underset{\sim}{T}$ to every subregion $\underset{\sim}{V}_n$ from the family M. Such correspondence is called the tensor field $\underset{\sim}{t}$ defined on the family M.

The tensor field $\underset{\sim}{t}$ is called finitely - valued if it assumes a finite, non - zero value on a finite number k of disjoint subregions $\underset{\sim}{V}_n$ from M and zero value on the remaining subregions from M.

The G-norm of the finitely - valued generalized stress field $\underset{\sim}{t}$ is defined by

$$|||\underset{\sim}{t}|||_G^2 = \int_{\underset{\sim}{V}} \underset{\sim}{t} \cdot \underset{\sim}{G}\,\underset{\sim}{t}\, d\underset{\sim}{x}\, e^{-t} dt \overset{df}{=} \sum_{n=1}^{k} \underset{\sim}{t}_n \cdot \underset{\sim}{G}\,\underset{\sim}{t}_n\, m(\underset{\sim}{V}_n) \qquad (5.9)$$

If follows from the definition of the finitely - valued field that the G - norm of such field always exists. Now we can construct the Hilbert space $\underset{\sim}{H}_1$ by the completion of the space of finitely - valued generalized stress fields with respect to G - norm. It can be shown that in our case the spaces $\underset{\sim}{H}$ and $\underset{\sim}{H}_1$ are equivalent.

The exact solution of the boundary problem can be approximated by the fields from arbitrary unitary space which was used to construct the Hilbert space. The space of smooth fields implies polynomial approximation while the space of finitely - valued fields implies one of the finite - element approximations.

It follows from the definition of the admissible field $\underset{\sim}{t}$ from the space $\underset{\sim}{H}$ that its value at a given point $(\underset{\sim}{x},t)$ in the region $\underset{\sim}{V}$ is not determined. Namely, two representatives of the class of equivalence $\underset{\sim}{t}$ may be different on the set which has the measure zero.

To determine the local properties of the admissible field we shall use the mean-value operator m_{ra} based on the finitely-valued field concept

$$m_{ra}\underset{\sim}{t}(\underset{\sim}{x},t) = \frac{1}{\alpha}\int_{t}^{t+a}\int_{R^3}\omega(\underset{\sim}{x}-\underset{\sim}{y})\,\underset{\sim}{t}(\underset{\sim}{y},\tau)\,d\underset{\sim}{y}\,e^{-\tau}\,d\tau \qquad (5.10)$$

where the field $\underset{\sim}{t}$ is assumed to vanish outside the region $\underset{\sim}{V}$, the function ω is defined by

$$\omega(x) = \begin{cases} 3/(4\Pi r^3) & \text{if } |x| < r \\ 0 & \text{if } |x| \geq r \end{cases} \qquad (5.11)$$

and

$$\alpha = \exp(-t) - \exp(-t-a) \qquad (5.12)$$

The mean-value operator m_{ra} determines the mean value of the admissible field $\underset{\sim}{t}$ in four-dimensional cylindrical neighbourhood of the point $(\underset{\sim}{x},t)$ defined by the radius r and the time interval a.

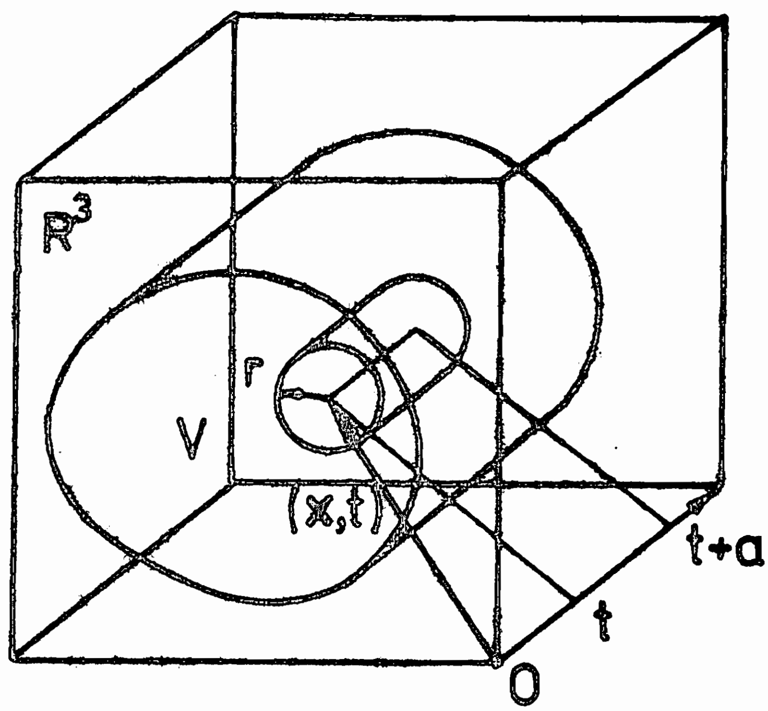

Fig. 12. Four-dimensional space-time cylinder

To define the derivatives of the admissible field we use the unitary space of smooth fields. Namely, the field $\underset{\sim}{d}$ is called the time derivative of the admissible field $\underset{\sim}{t}$ if

$$<\underset{\sim}{t},\dot{\underset{\sim}{t}}^*>_G = -<\underset{\sim}{d},\underset{\sim}{t}^*>_G + <\underset{\sim}{t},\underset{\sim}{t}^*>_G \quad \text{for every} \quad t^* \in \underset{\sim}{C}^\infty(\underset{\sim}{V}) \qquad (5.13)$$

The vector field $\underset{\sim}{c}$ is called the divergence of the tensor field $\underset{\sim}{\tau}$ if the relation

$$<\underset{\sim}{\tau}, \text{grad } \underset{\sim}{v}> = - <\underset{\sim}{c},\underset{\sim}{v}> \qquad (5.14)$$

is satisfied for every smooth vector field $\underset{\sim}{v}$ vanishing on the boundary B.

5.2 Kinematically and statically admissible fields

We define the subspace $\underset{\sim}{K}$ of generalized kinematically admissible fields as the completion of the unitary subspace of all smooth kinematically admissible fields with respect to the G-norm. It follows from the above definition that every field $\underset{\sim}{m}$ from $\underset{\sim}{K}$ satisfies the compatibility equation, as all terms of the Cauchy sequence are compatible.

Consequently the subspace $\underset{\sim}{S}$ of generalized statically admissible fields is defined as the completion of the unitary subspace of smooth statically admissible fields with respect to the G-norm. Since every field of the Cauchy sequence is in equilibrium then every field form $\underset{\sim}{S}$ satisfies the equilibrium equation.

It follows from the equation (4.6) that the space $\underset{\sim}{H}$ is the orthogonal sum of the subspaces $\underset{\sim}{K}$ and $\underset{\sim}{S}$, i. e. every admissible field $\underset{\sim}{t}$ from $\underset{\sim}{H}$ can be uniquely decomposed into the sum of kinematically admissible field $\underset{\sim}{m}$ and the statically admissible field $\underset{\sim}{r}$.

$$\underset{\sim}{H} = \underset{\sim}{S} \oplus \underset{\sim}{K} \qquad (5.15)$$

Geometrical interpretation of the orthogonal decomposition of the space $\underset{\sim}{H}$ is presented in Fig. 13. The horizontal plane represents the infinite-dimensional subspace $\underset{\sim}{K}$ of kinematically admissible fields, while the vertical axis represents the orthogonal subspace $\underset{\sim}{S}$ of statically admissible fields. The vector $\underset{\sim}{t}$ represents the admissible generalized stress field.

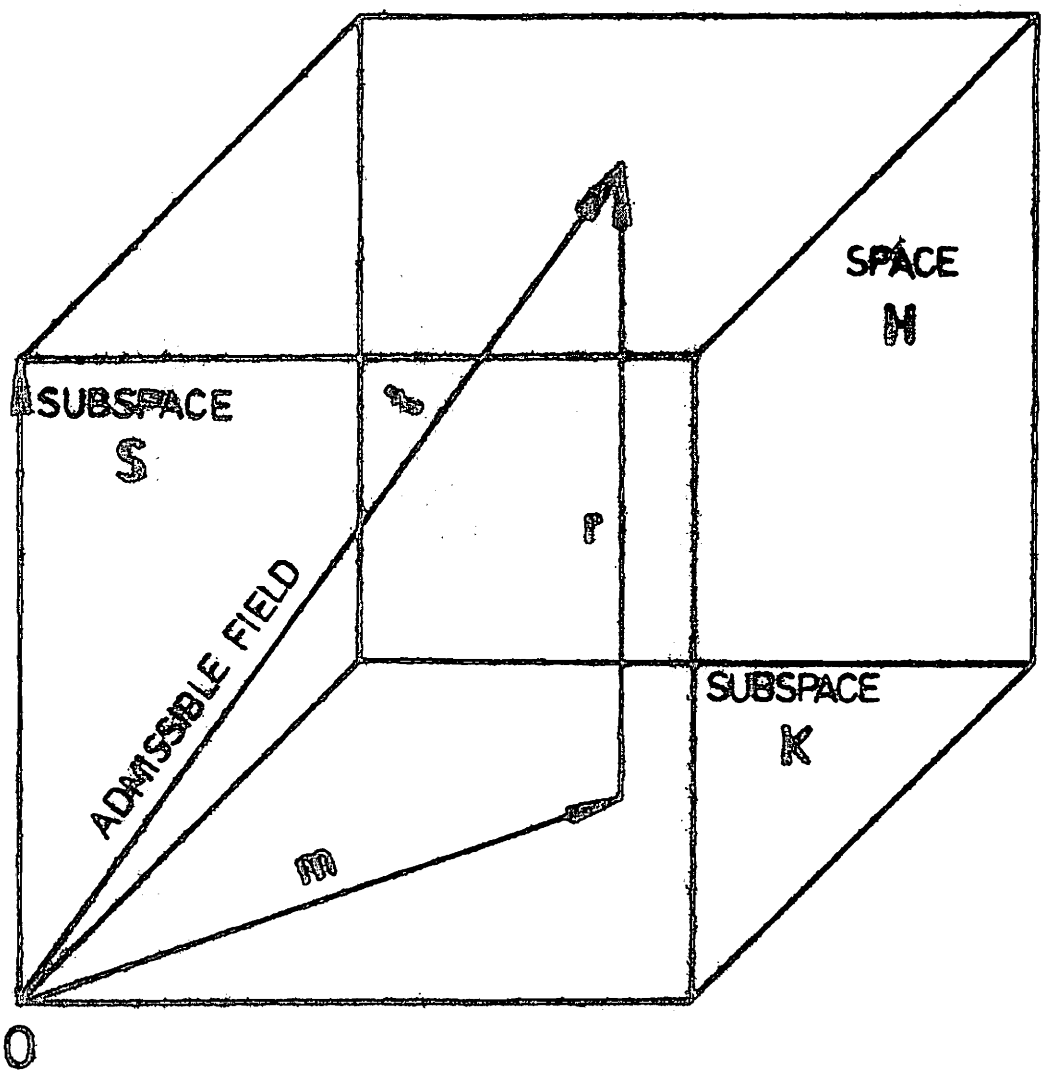

Fig. 13. Orthogonal decomposition of the space $\underset{\sim}{H}$

The components $\underset{\sim}{r}$ and $\underset{\sim}{m}$ of the field $\underset{\sim}{t}$ can be obtained by G-orthogonal projection of the field $\underset{\sim}{t}$ onto the subspaces $\underset{\sim}{S}$ and $\underset{\sim}{K}$, respectively.

5.3 Time-differentiable admissible field

The generalized stress field $\underset{\sim}{t}$ is called time differentiable if there exists the admissible field $\underset{\sim}{d}$ such that

$$\underset{\sim}{t}(\underset{\sim}{x},t) = \int_0^t \underset{\sim}{d}(\underset{\sim}{x},t')\,dt' \qquad (5.16)$$

where the time integral is defined as the limit

$$\int_0^t \underset{\sim}{d}(\underset{\sim}{x},t')\,dt' = \lim_{n\to\infty} \int_0^t \underset{\sim}{d}_{(n)}(\underset{\sim}{x},t')\,dt' \qquad (5.17)$$

where $\{\underset{\sim}{d}_{(n)}\}$ is the sequence of smooth fields convergent to $\underset{\sim}{d}$.

Making use of the equation (4.12) and the Schwarz inequality we obtain for every smooth generalized stress field $\underset{\sim}{t}$ vanishing at $t = 0$

$$|||\,\underset{\sim}{t}\,|||_G \le 2\,|||\,\dot{\underset{\sim}{t}}\,|||_G \qquad (5.18)$$

Hence every time differentiable field is admissible.

We shall denote by $\underset{\sim}{H}'$, $\underset{\sim}{K}'$ and $\underset{\sim}{S}'$ the subspaces of all fields from $\underset{\sim}{H}$, $\underset{\sim}{K}$ and $\underset{\sim}{S}$, respectively, which are time-differentiable. We have $\underset{\sim}{H}' \subset \underset{\sim}{H}$, $\underset{\sim}{K}' \subset \underset{\sim}{K}$ and $\underset{\sim}{S}' \subset \underset{\sim}{S}$. The properties resulting from differentiation with respect to coordinates x_i are preserved in the new subspaces. Namely, if the field $\underset{\sim}{t}$ from $\underset{\sim}{H}$ is G-orthogonal to the subspace $\underset{\sim}{S}'$ then the field $\underset{\sim}{t}$ belongs to $\underset{\sim}{K}'$. We shall use this property to construct the generalized strain field in the space $\underset{\sim}{H}$.

5.4 Global plastic potential

To define the global plastic potential we introduce the auxiliary function $\varphi_c(\underset{\sim}{t})$ defined in the space $\underset{\sim}{T}$ of generalized tensors by

$$\varphi_c(\underset{\sim}{t}) = \begin{cases} \varphi(\underset{\sim}{t}) & \text{if } \varphi(\underset{\sim}{t}) < +\infty \\ c & \text{if } \varphi(\underset{\sim}{t}) = +\infty \end{cases} \tag{5.19}$$

where c is positive real number. Now the global plastic potential of the field $\underset{\sim}{t}$, which is the limit of Cauchy sequence $\{\underset{\sim}{t}_{(n)}\}$ is defined as

$$\Phi(\underset{\sim}{t}) = \lim_{c\to\infty} \lim_{n\to\infty} \int_{\underset{\sim}{V}} \varphi_c(\underset{\sim}{t}_{(n)}(\underset{\sim}{x};t))\, d\underset{\sim}{x}\, e^{-t}\, dt \tag{5.20}$$

It follows from the properties of φ that the plastic potential $\Phi(\underset{\sim}{t})$ is lower-semicontinuous, convex and it attains the minimum at the origin of the space $\underset{\sim}{H}$.

The plastic flow law is formulated in the space $\underset{\sim}{H}$ by the relation

$$\underset{\sim}{G}^{-1}\dot{\underset{\sim}{e}}^{p} \in \partial\Phi(\underset{\sim}{s}) \tag{5.21}$$

which is equivalent to the requirement

$$\langle \dot{\underset{\sim}{e}}^{p}, \underset{\sim}{s} \rangle - \Phi(\underset{\sim}{s}) = \Phi^*(\underset{\sim}{G}^{-1}\dot{\underset{\sim}{e}}^{p}) \tag{5.22}$$

where the dual plastic potential Φ^* is defined by

$$\Phi^*(\underset{\sim}{G}^{-1}\dot{\underset{\sim}{e}}^{p}) = \sup_{\underset{\sim}{t}\in\underset{\sim}{H}} [\langle \dot{\underset{\sim}{e}}^{p}, \underset{\sim}{t} \rangle - \Phi(\underset{\sim}{t})] \tag{5.23}$$

The above definition of subdifferential in the space $\underset{\sim}{H}$ is based on G-scalar product in the stress space. Alternatively we would consider the dual strain space, which is obtained by multiplication of stress fields by matrix $\underset{\sim}{G}$

$$\langle \underset{\sim}{G}^{-1}\dot{\underset{\sim}{e}}^{p}, \underset{\sim}{s} \rangle_G = \langle \dot{\underset{\sim}{e}}^{p}, \underset{\sim}{s} \rangle \tag{5.24}$$

We can show, that the plastic flow law established in the space $\underset{\sim}{H}$ is compatible with the formulation given for a space-time element of material. Namely, for every point $(\underset{\sim}{x}_o, t_o) \in \underset{\sim}{V}$ we can construct space-time cylindrical neighbourhood $\underset{\sim}{V}_o$ of radius r and length a, which is enclosed in the region $\underset{\sim}{V}$. Then we introduce the finitely-valued stress fields $\underset{\sim}{t}$ and $\underset{\sim}{s}$ which assume the non-zero values $\underset{\sim}{t}_o$ and $\underset{\sim}{s}_o$, respectively, in the subregion $\underset{\sim}{V}_o$ only. Substituting such fields to the plastic flow law formulated in the space $\underset{\sim}{H}$ we obtain the relation

$$m_{ra}\, \dot{\underset{\sim}{e}}^p(\underset{\sim}{x}_o, t_o) \cdot \underset{\sim}{s}_o - \varphi(\underset{\sim}{s}_o) = \sup_{\underset{\sim}{t}_o \in \underset{\sim}{T}} [m_{ra} \dot{\underset{\sim}{e}}^p(\underset{\sim}{x}_o, t_o) \cdot \underset{\sim}{t}_o - \varphi(\underset{\sim}{t}_o)] \qquad (5.25)$$

which expresses the plastic flow law formulated in section 3.3.

5.5 Minimum principle

The initial-boundary value problem is formulated in the space of admissible fields $\underset{\sim}{H}$ as follows:

> Find the field $\underset{\sim}{r}$ from $\underset{\sim}{S}'$ and the field $\underset{\sim}{m}$ from $\underset{\sim}{K}'$ such that
>
> $$\dot{\underset{\sim}{r}} + \dot{\underset{\sim}{m}} \in \partial\Phi(\underset{\sim}{s}^o - \underset{\sim}{r}) \qquad (5.26)$$
>
> where the perfectly elastic solution $\underset{\sim}{s}^o$ is given

The minimum principle follows directly from the definition of sub-differential. Namely, for every $\underset{\sim}{r}$ from $\underset{\sim}{S}'$ and $\underset{\sim}{m}$ from $\underset{\sim}{K}'$ we have

$$\Lambda(\underset{\sim}{r}, \underset{\sim}{m}) = \Phi(\underset{\sim}{s}^o - \underset{\sim}{r}) - \langle \underset{\sim}{s}^o - \underset{\sim}{r}, \dot{\underset{\sim}{r}} + \dot{\underset{\sim}{m}} \rangle_G + \Phi^*(\dot{\underset{\sim}{r}} + \dot{\underset{\sim}{m}}) \geq 0 \qquad (5.27)$$

The functional Λ is convex and it attains the minimum (equal to zero) if the plastic flow law is satisfied.

Let us consider the functional Λ_o defined in the subspace $\underset{\sim}{S}'$ by

$$\Lambda_o(\underset{\sim}{r}) = \Phi_o(\underset{\sim}{s}^o - \underset{\sim}{r}) - \langle \underset{\sim}{s}^o - \underset{\sim}{r}, \dot{\underset{\sim}{r}} \rangle_G + \Phi_o^*(\dot{\underset{\sim}{r}}) \tag{5.28}$$

where $\Phi_o(\underset{\sim}{t}) = \Phi(\underset{\sim}{t})$ for all fields $\underset{\sim}{t}$ from the <u>plane</u> $\underset{\sim}{P} = \underset{\sim}{s}^o + \underset{\sim}{S}'$ and $\underset{\sim}{\Phi}_o^*$ is the polar potential defined by

$$\Phi_o^*(\dot{\underset{\sim}{r}}) = \sup_{\underset{\sim}{r}^* \in \underset{\sim}{S}'} [\langle \underset{\sim}{s}^o - \underset{\sim}{r}^*, \dot{\underset{\sim}{r}} \rangle_G - \Phi_o(\underset{\sim}{s}^o - \underset{\sim}{r}^*)] \tag{5.29}$$

It follows from the equation (4.12) that the functional Λ_o is <u>strictly convex</u>. Hence it attains an absolute minimum (equal to zero) at the unique field $\underset{\sim}{r}_o$. This fact assures <u>uniqueness of the stress field</u>.

Making use of the inequality

$$\sup_{\underset{\sim}{m}^* \in \underset{\sim}{K}} [\langle \underset{\sim}{m}^*, \dot{\underset{\sim}{m}} \rangle_G - \Phi(\underset{\sim}{s}^o - \underset{\sim}{r}^* + \underset{\sim}{m}^*)] \geq - \Phi(\underset{\sim}{s}^o - \underset{\sim}{r}^*) \tag{5.30}$$

and the G-orthogonality of fields from $\underset{\sim}{K}$ and $\underset{\sim}{S}$ we obtain

$$\Lambda(\underset{\sim}{r}, \underset{\sim}{m}) \geq \Lambda_o(\underset{\sim}{r}) \tag{5.31}$$

in the plane $\underset{\sim}{P}$.

Now we can establish the MINIMUM PRINCIPLE for the residual stress fields.

> The functional $\Lambda_o(\underset{\sim}{r}^*)$ defined for prescribed perfectly elastic solution $\underset{\sim}{s}^o$ and for all differentiable statically admissible generalized stress field $\underset{\sim}{r}^*$ attains an absolute minimum (equal to zero) if and only if $\underset{\sim}{r}^*$ is equal to the actual generalized residual stress field $\underset{\sim}{r}$.

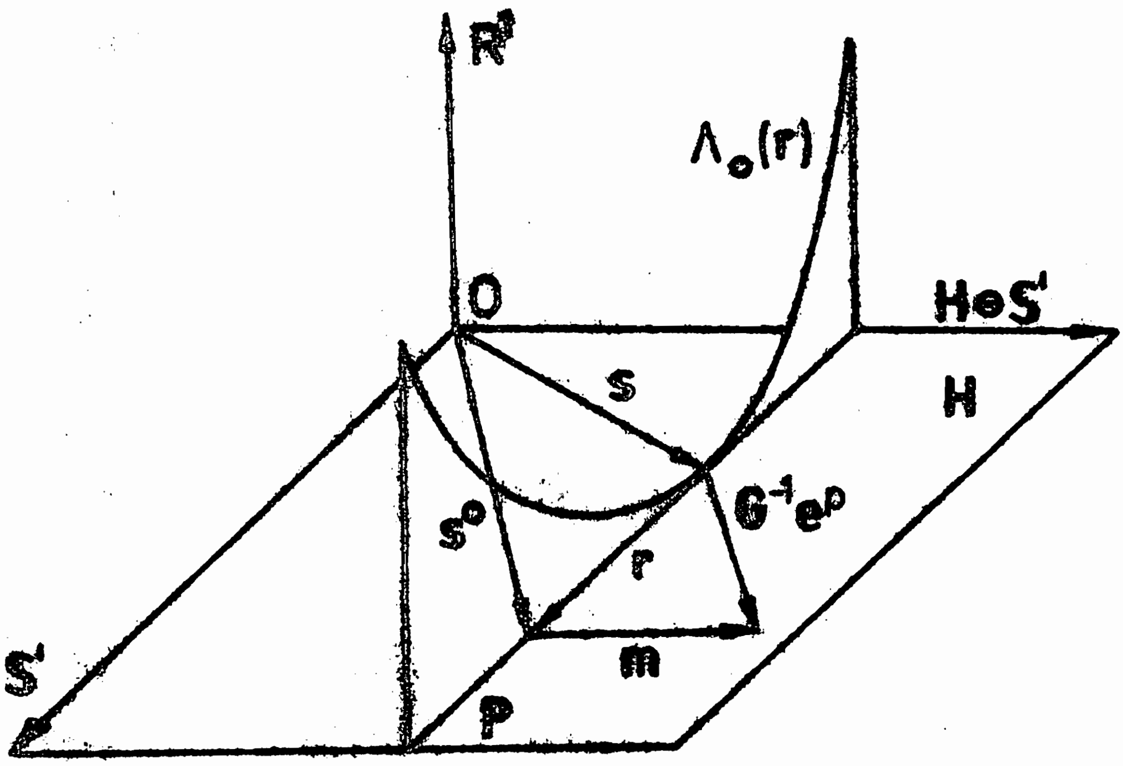

Fig. 14. Minimum principle for residual stress

The construction of the kinematically admissible generalized field $\dot{\underset{\sim}{m}}$ is not always unique. Hence the <u>generalized strain field may be not unique</u>.

Taking into account the property of the regular standard model we conclude that in this case the strain field $\underset{\sim}{\varepsilon}$ is always uniquely determined. Hence <u>for the regular standard material the solution of the initial-boundary value problem is unique</u>.

It follows from the above consideration that the solution exists if the plastic potential $\underset{\sim}{\Phi}$ in the plane $\underset{\sim}{P}$ assumes at least one finite value. Hence we establish the <u>necessary and sufficient condition for the existence of solution</u>:

The intersection $\underset{\sim}{E}^o$ of the plane $\underset{\sim}{P} = \underset{\sim}{s}^o + \underset{\sim}{S}'$ and the effective domain $\underset{\sim}{E}$ of the potential $\underset{\sim}{\Phi}$ is not empty

where the effective domain of the potential $\underset{\sim}{\Phi}$ in the space $\underset{\sim}{H}$ is defined by

$$\underset{\sim}{E} = [\underset{\sim}{t} : \Phi(\underset{\sim}{t}) < +\infty] \tag{5.32}$$

5.6 Minimum principles for the elastic-plastic model

In the particular case of the generalized standard elastic-plastic material we can express the minimum principle in terms of the elastic region $\underset{\sim}{E}$ which is defined in the space $\underset{\sim}{H}$ by

$$\underset{\sim}{E} = [\underset{\sim}{t} : \Phi(\underset{\sim}{t}) = 0] \tag{5.33}$$

We introduce the yield function determined in the space $\underset{\sim}{H}$ by the elastic region $\underset{\sim}{E}$

$$F(\underset{\sim}{t}) = \inf [c > 0 : \underset{\sim}{t}/c \in \underset{\sim}{E}] \tag{5.34}$$

We construct in the subspace $\underset{\sim}{S}'$ the set $\underset{\sim}{U}$ determined by the perfectly elastic solution $\underset{\sim}{s}^{o}$ and the intersection $\underset{\sim}{E}^{o}$.

$$\underset{\sim}{U} = [\underset{\sim}{t} : \langle \underset{\sim}{s}^{o} - \underset{\sim}{t}, \dot{\underset{\sim}{t}} \rangle_G \geq \Phi^*_o(\dot{\underset{\sim}{t}})] \tag{5.35}$$

Where the polar potential Φ^*_o in the subspace $\underset{\sim}{S}'$ is defined by

$$\Phi^*_o(\dot{t}) = \sup_{\underset{\sim}{t}^* \in \underset{\sim}{E}^{o}} \langle \underset{\sim}{s}^{o} - \underset{\sim}{t}^*, \dot{\underset{\sim}{t}} \rangle_G \tag{5.36}$$

It follows from the construction that the closed set $\underset{\sim}{U}$ is strictly convex and contains origin on its boundary.

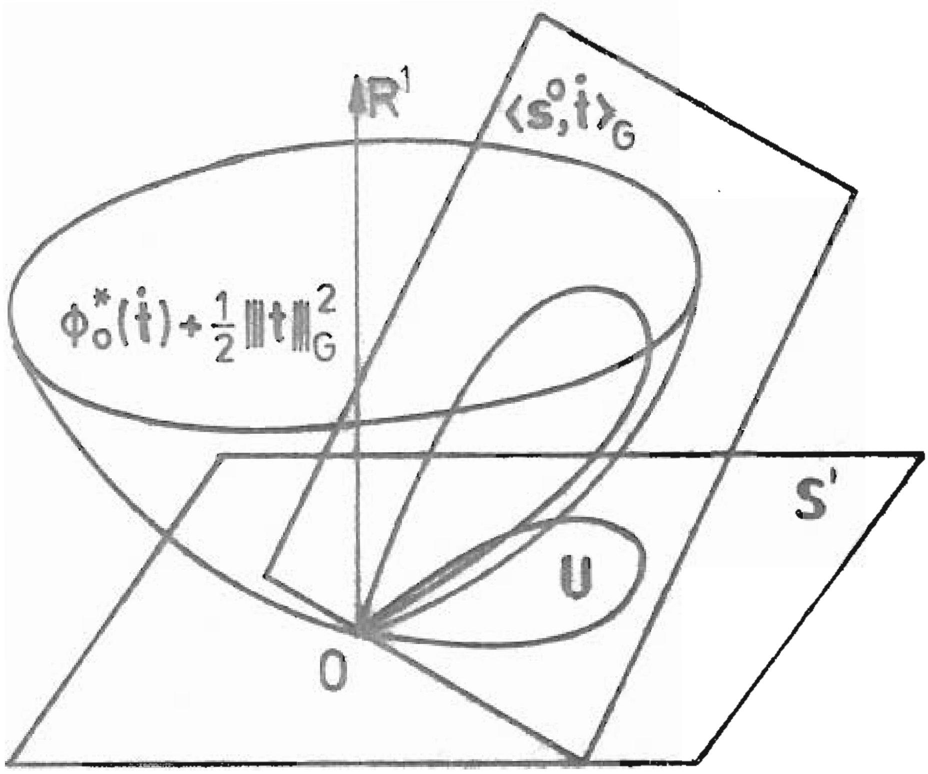

Fig. 15. Construction of the set $\underset{\sim}{U}$

We also introduce the functional W defined in the space $\underset{\sim}{S}'$ determined by the region $\underset{\sim}{U}$

$$W(\underset{\sim}{t}) = \inf [c > 0 : \underset{\sim}{t}/c \in \underset{\sim}{U}] \qquad (5.37)$$

The complementary principles for the stress field are based on the fact that the sets $\underset{\sim}{s}^o - \underset{\sim}{U}$ and $\underset{\sim}{E}^o$ have only one common field $\underset{\sim}{s}$, which is the solution of the problem. Thus we have

MINIMUM PRINCIPLE for residual stress

The yield function $F(\underset{\sim}{s}^o - \underset{\sim}{r}^*)$ defined for prescribed perfectly elastic solution $\underset{\sim}{s}^o$ and for all fields $\underset{\sim}{r}^*$ from $\underset{\sim}{U}$ attains an absolute minimum (equal to 1) if and only if $\underset{\sim}{r}^*$ is equal to the actual generalized residual stress field $\underset{\sim}{r}$.

and MINIMUM PRINCIPLE for stress

The functional $W(\underset{\sim}{s}^{o} - \underset{\sim}{s}^{*})$ defined for prescribed perfectly elastic solution $\underset{\sim}{s}^{o}$ and for all fields $\underset{\sim}{s}^{*}$ from $\underset{\sim}{E}^{o}$ attains an absolute minimum (equal to 1) if only if $\underset{\sim}{s}^{*}$ is equal to the actual generalized stress field $\underset{\sim}{s}$.

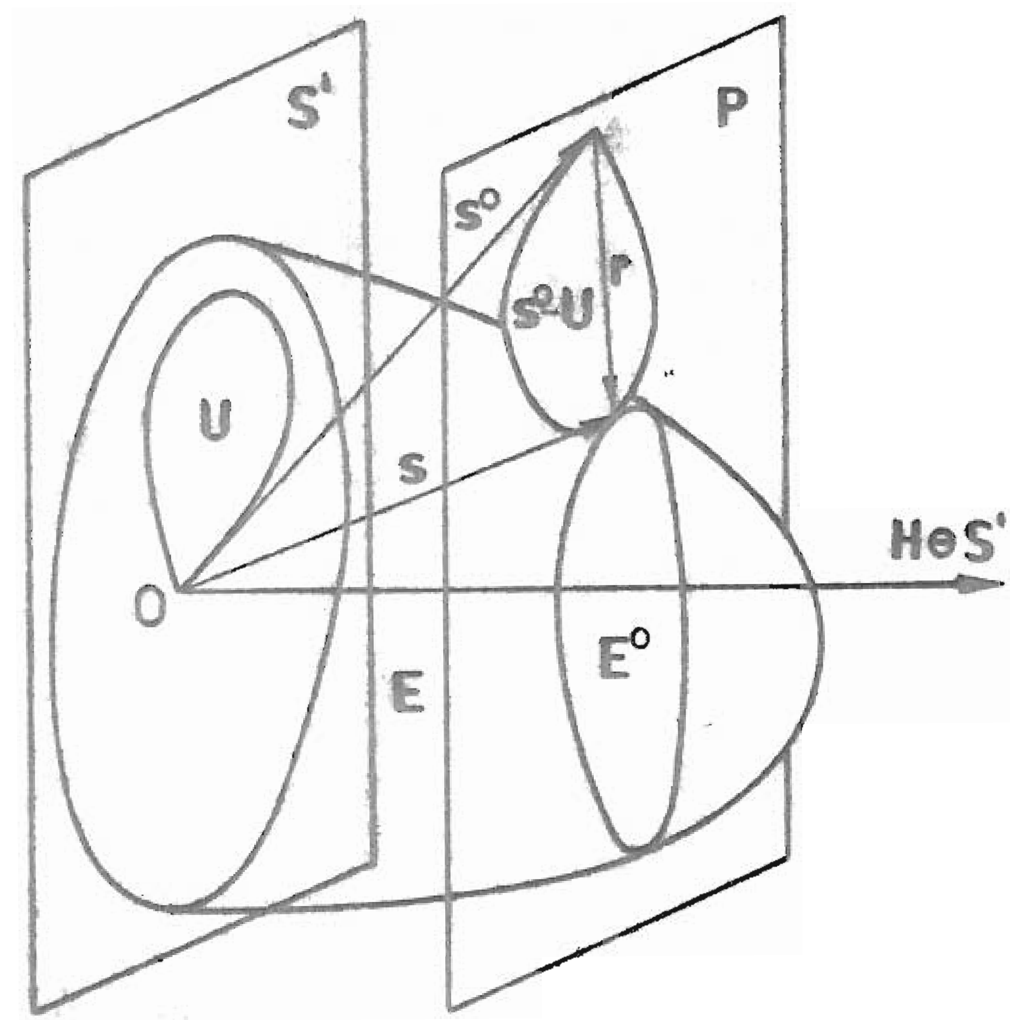

Fig. 16. Construction of the stress field for elastic-plastic material

5.7 Other constructions of the scalar product

The space-time scalar product based on the function e^{-t} is not unique. We can repeat our derivations for the scalar product

$$<\underset{\sim}{t},\underset{\sim}{t}^*>_G = \int_{\underset{\sim}{V}} \underset{\sim}{t} \cdot \underset{\sim}{G}\,\underset{\sim}{t}^* \, d\underset{\sim}{x}\, h(t)\, dt \tag{5.38}$$

provided that the non-negative function $h(t)$ defined for $t \geq 0$ assures the existence of the measure of the region $\underset{\sim}{V}$

$$m(\underset{\sim}{V}) = |V| \int_0^{\infty} h(t)\, dt < +\infty \tag{5.39}$$

and assures that the product $<\underset{\sim}{t},\dot{\underset{\sim}{t}}>$ is strictly convex in the space $\underset{\sim}{H}$. The scalar product based on function

$$h_1(t) = \begin{cases} 1 & t \leq t_o \\ 0 & t > t_o \end{cases} \tag{5.40}$$

does not assure the second requirement as

$$2<\underset{\sim}{t},\dot{\underset{\sim}{t}}>_G = \int_V \underset{\sim}{t} \cdot \underset{\sim}{G}\,\underset{\sim}{t}\, d\underset{\sim}{x} \Big|_{t = t_o}$$

If we introduce the scalar product based on the function $h_2(t) = h_1(t)e^{-t}$ then we have

$$2<\underset{\sim}{t},\dot{\underset{\sim}{t}}>_G = <\underset{\sim}{t},\underset{\sim}{t}>_G + e^{-t_o} \int_V \underset{\sim}{t} \cdot G\,\underset{\sim}{t}\, d\underset{\sim}{x} \Big|_{t = t_o}$$

and all requirements are satisfied.

6. SOLUTION OF THE RATE BOUNDARY VALUE PROBLEM

6.1 Construction of the space of admissible fields

To construct the space of admissible fields H we shall use the space $C^{\infty}(V)$ of all smooth generalized tensor fields $\underset{\sim}{t}$, defined in three-dimensional region V at time t_o, as a unitary space provided with the scalar product

$$(\underset{\sim}{t},\underset{\sim}{t}^*)_G = \int_V \underset{\sim}{t} \cdot \underset{\sim}{G}\,\underset{\sim}{t}^*\, d\underset{\sim}{x} \tag{6.1}$$

The space H is defined as the completion of the unitary space.

To determine the local properties of the admissible fields from H we shall use the concept of fields from alternative unitary space of all finitely-valued fields defined on the Lebesque-measurable family of subsets $V_n \subset V$

$$m_r \underset{\sim}{t}(\underset{\sim}{x},t_o) = \int_V \omega(\underset{\sim}{x}-\underset{\sim}{y})\ \underset{\sim}{t}(\underset{\sim}{y},t_o)\, d\underset{\sim}{y} \tag{6.2}$$

The subspace K of generalized kinematically admissible fields is defined as the completion of the subspace of all smooth kinematically admissible fields with respect to the G-norm.

Consequently the subspace S of generalized statically admissible fields is obtained by completion of the unitary subspace of smooth statically admissible fields.

The plastic potential Φ of the field $\underset{\sim}{t} = \lim_{n\to\infty} \underset{\sim}{t}_{(n)}$ from the space H is defined by

$$\Phi(\underset{\sim}{t}) = \lim_{c\to\infty} \lim_{n\to\infty} \int_V \varphi_c(\underset{\sim}{t}_{(n)}(\underset{\sim}{x},t_o))\ d\underset{\sim}{x} \tag{6.3}$$

The plastic flow law in the space H has the form

$$\underset{\sim}{G}^{-1}\dot{\underset{\sim}{e}}^p \in \partial\Phi(\underset{\sim}{s}) \tag{6.4}$$

where the subdifferential is determined by prescribed generalized stress field $\underset{\sim}{s}$.

Analogously to the considerations in section 5.4 we can show that the above plastic flow law is compatible with the primary formulation for the small element of the body.

6.2 Minimum principle

To formulate the rate boundary value problem in the space H we introduce the functional χ which is defined in the space H as the indicator of subdifferential $\partial\Phi(\underset{\sim}{s})$

$$\chi(\underset{\sim}{t}) = \begin{cases} 0 & \text{if} \quad \underset{\sim}{t} \in \partial\Phi(\underset{\sim}{s}) \\ +\infty & \text{if} \quad \underset{\sim}{t} \notin \partial\Phi(\underset{\sim}{s}) \end{cases} \tag{6.5}$$

The polar functional $\chi^*(\dot{\underset{\sim}{s}})$ represents the rate of plastic potential Φ due to the stress rate $\dot{\underset{\sim}{s}}$

$$\chi^*(\dot{\underset{\sim}{s}}) = \sup_{\underset{\sim}{t} \in H} [\, (\dot{\underset{\sim}{s}},\underset{\sim}{t})_G - \chi(\underset{\sim}{t})] \tag{6.6}$$

The rate boundary value problem can be formulated as follows

Find the field $\dot{\underset{\sim}{r}}$ from S and the field $\dot{\underset{\sim}{m}}$ from K such that $\dot{\underset{\sim}{r}} + \dot{\underset{\sim}{m}} \in \partial\chi^*(\dot{\underset{\sim}{s}}^o - \dot{\underset{\sim}{r}})$ (6.7) where the rate $\dot{\underset{\sim}{s}}^o$ of the perfectly elastic solution is given

Indeed, the above relation written in the form

$$\chi^*(\dot{\underset{\sim}{s}}^o - \dot{\underset{\sim}{r}}) - (\dot{\underset{\sim}{s}}^o - \dot{\underset{\sim}{r}}, \dot{\underset{\sim}{r}} + \dot{\underset{\sim}{m}})_G + \chi(\dot{\underset{\sim}{r}} + \dot{\underset{\sim}{m}}) = 0 \tag{6.8}$$

is directly equivalent to the plastic flow law at the moment t_o

$$\dot{\underset{\sim}{r}} + \dot{\underset{\sim}{m}} \in \partial\Phi(\underset{\sim}{s}) \tag{6.9}$$

and the compatibility of the rate of plastic potential with the rate of stress field

$$(\dot{\underset{\sim}{s}}, \dot{\underset{\sim}{e}}^p) = \chi^*(\dot{\underset{\sim}{s}}) \tag{6.10}$$

Now, repeating the derivations from the previous chapter we introduce the functional Λ_o defined in the subspace S of statically admissible fields

$$\Lambda_o(\dot{\underset{\sim}{r}}) = \chi_o^*(\dot{\underset{\sim}{s}}^o - \dot{\underset{\sim}{r}}) - (\dot{\underset{\sim}{s}}^o - \dot{\underset{\sim}{r}}, \dot{\underset{\sim}{r}})_G + \chi_o(\dot{\underset{\sim}{r}}) \tag{6.11}$$

where $\chi_o^*(\underset{\sim}{t}) = \chi^*(\underset{\sim}{t})$ for all fields $\underset{\sim}{t}$ from the plane $P = \dot{\underset{\sim}{s}}^o + S$ and χ_o is its polar potential defined by

$$\chi_o(\dot{\underset{\sim}{r}}) = \sup_{\dot{\underset{\sim}{r}}^* \in S} [(\dot{\underset{\sim}{s}}^o - \dot{\underset{\sim}{r}}^*, \dot{\underset{\sim}{r}})_G - \chi_o^*(\dot{\underset{\sim}{s}}^o - \dot{\underset{\sim}{r}}^*)] \tag{6.12}$$

Since the functional Λ_o is strictly convex in P then the uniqueness of the generalized stress rate $\dot{\underset{\sim}{s}}$ is assured. The construction of the kinematically admissible field $\dot{\underset{\sim}{m}}$ resulting from the minimization of the functional

$$\Lambda(\dot{\underset{\sim}{r}}, \dot{\underset{\sim}{m}}) = \chi^*(\dot{\underset{\sim}{s}}^o - \dot{\underset{\sim}{r}}) - (\dot{\underset{\sim}{s}}^o - \dot{\underset{\sim}{r}}, \dot{\underset{\sim}{r}} + \dot{\underset{\sim}{m}})_G + \chi(\dot{\underset{\sim}{r}} + \dot{\underset{\sim}{m}}) \tag{6.13}$$

in the space H may be not unique. Hence the rate of generalized plastic strain $\dot{\underset{\sim}{e}}^p$ is, in general, not unique. Uniqueness of strains is assured in the case of regular materials.

The MINIMUM PRINCIPLE for the rate of generalized residual stress field takes the form

The functional $\Lambda_o(\dot{\underset{\sim}{r}}^*)$ defined for the prescribed perfectly elastic solution $\underset{\sim}{s}^o$ and its rate $\dot{\underset{\sim}{s}}^o$ and for all statically admissible fields $\dot{\underset{\sim}{r}}^*$ attains an absolute minimum (equal to zero) if and only if $\dot{\underset{\sim}{r}}^*$ is equal to the actual rate $\dot{\underset{\sim}{r}}$ of generalized residual stress field.

The necessary and sufficient condition for the existence of solution can be presented in the form

The intersection A^o of the plane $P = \dot{\underset{\sim}{s}}^o + S$ and the effective domain A of the potential χ^* is not empty.

6.3 Minimum principle for the elastic-plastic model

In the particular case of the generalized standard elastic-plastic material the subdifferential $\partial\Phi(\underset{\sim}{s})$ is a <u>cone</u> in the space H. Then the <u>effective domain</u> A of the potential χ^* is identical with the <u>polar cone</u> $[\partial\Phi(\underset{\sim}{s})]^*$, i. e.

$$A = [\underset{\sim}{t} : (\underset{\sim}{t},\underset{\sim}{t}^*)_G \leq 0 \quad \text{for all} \quad \underset{\sim}{t}^* \in \partial\Phi(\underset{\sim}{s})] \quad . \tag{6.14}$$

and the potential χ^* is the <u>indicator</u> of the domain A. The domain A represents the set of all <u>plastically admissible</u> generalized stress rates.

Now the minimum principle is equivalent to the minimization of functional

$$\Lambda_o(\dot{\underset{\sim}{r}}) = \sup_{\dot{\underset{\sim}{r}}^* \in \dot{\underset{\sim}{s}}^o - A^o} (\dot{\underset{\sim}{r}} - \dot{\underset{\sim}{r}}^*, \dot{\underset{\sim}{r}})_G \tag{6.15}$$

in the domain $\dot{\underset{\sim}{s}}^o - A^o$. It follows from the convexity of the region A^o that Λ_o reaches minimum at the unique field $\dot{\underset{\sim}{r}}$, which represents the shortest distance between the field $\dot{\underset{\sim}{s}}^o$ and the domain A^o in terms of the G-norm.

The geometrical representation of the mathematical objects discussed above is given in Fig. 17. Here the <u>elastic region</u> E is the effective domain of the plastic potential Φ and the <u>yield surface</u> Y is the boundary of E. The intersection of plane P and the region E is denoted by E^o.

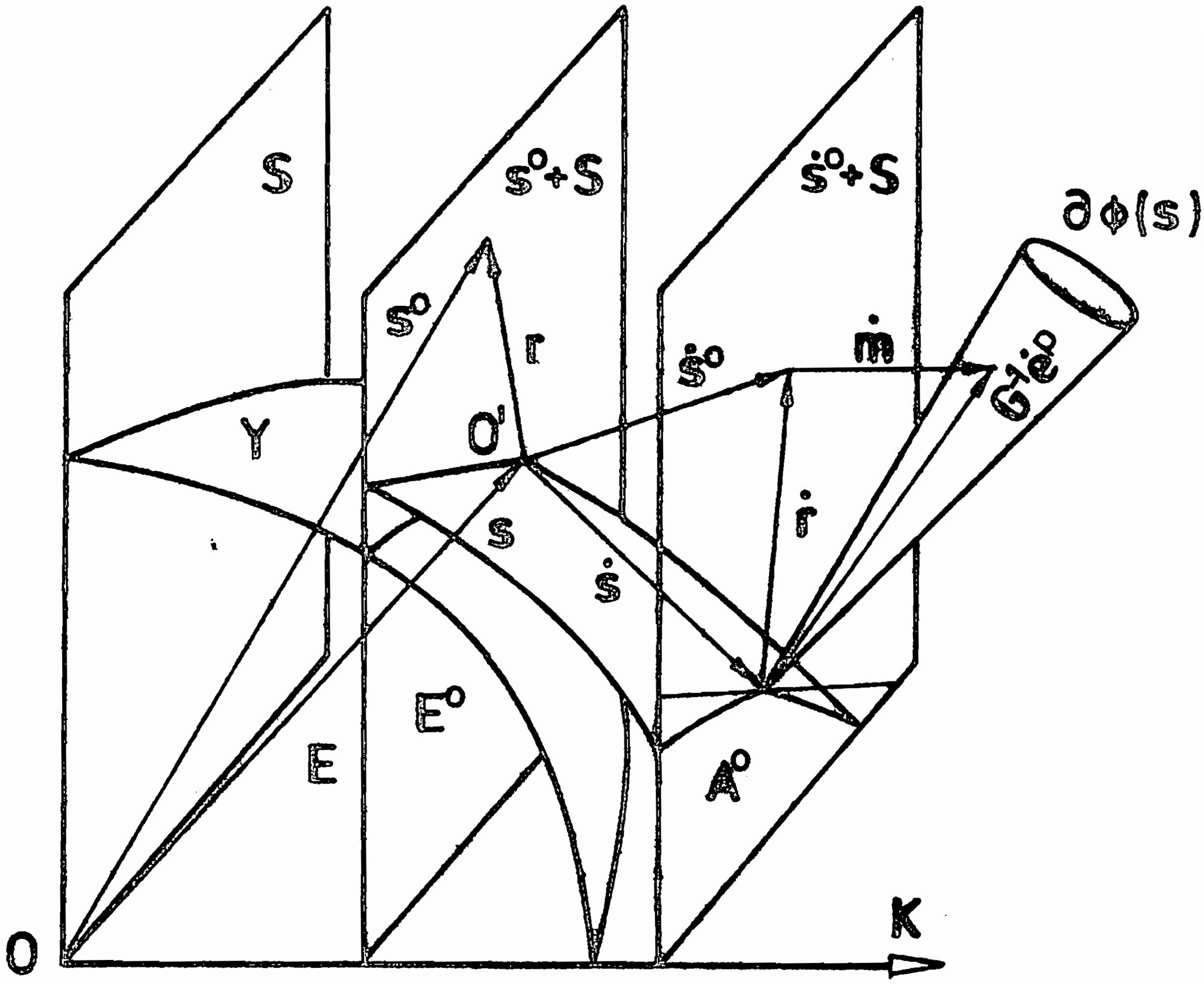

Fig. 17. Construction of the solution in space $\underset{\sim}{H}$.

Hence we can establish the MINIMUM PRINCIPLE for the stress rate

The functional $\| \underset{\sim}{\dot{s}}^{o} - \underset{\sim}{\dot{s}}^{*} \|_G$ defined for the prescribed rate $\underset{\sim}{\dot{s}}^{o}$ of the perfectly elastic solution and for all stress fields $\underset{\sim}{\dot{s}}^{*}$ from the domain A^{o} attains an absolute minimum if and only if $\underset{\sim}{\dot{s}}^{*}$ is equal to the actual rate $\underset{\sim}{\dot{s}}$ of generalized stress field

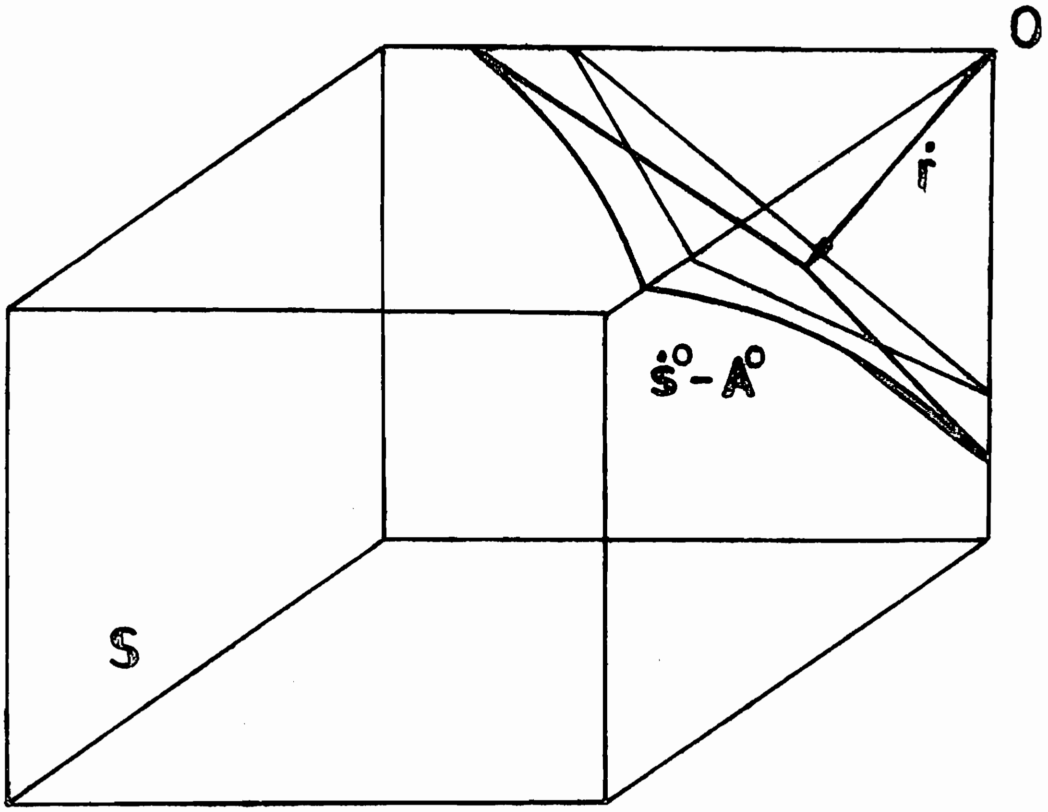

Fig. 18. Construction of the stress rate field for elastic-plastic material

6.4 Minimum principle for the regular point of plastic potential

We shall call the stress field $\underset{\sim}{s}$ the regular point of the plastic potential Φ if the subdifferential $\partial\Phi(\underset{\sim}{s})$ consists of one field $\dot{\underset{\sim}{g}}$ only. In this case we obtain directly the fields $\dot{\underset{\sim}{r}}$ and $\dot{\underset{\sim}{m}}$ by G-orthogonal decomposition of the field $\dot{\underset{\sim}{g}}$ into statically admissible and kinematically admissible component

$$\dot{\underset{\sim}{r}} + \dot{\underset{\sim}{m}} = \dot{\underset{\sim}{g}} \qquad (6.16)$$

Since such decomposition is unique we obtain unique rates of generalized strain and stress fields.

Hence, if $\partial\Phi(\underset{\sim}{s}) = \underset{\sim}{\dot{g}}$ then we have the MINIMUM PRINCIPLE for residual stress

> The functional $\| \underset{\sim}{\dot{g}} - \underset{\sim}{\dot{r}}^* \|_G$ defined for the prescribed field $\underset{\sim}{\dot{g}}$ and for all fields $\underset{\sim}{\dot{r}}$ form S attains an absolute minimum if and if $\underset{\sim}{\dot{r}}^*$ is equal to the actual rate $\underset{\sim}{\dot{r}}$ of generalized residual stress field.

and the complementary MINIMUM PRINCIPLE

> The functional $\| \underset{\sim}{\dot{g}} - \underset{\sim}{\dot{m}}^* \|_G$ defined for the prescribed field $\underset{\sim}{\dot{g}}$ and for all fields $\underset{\sim}{\dot{m}}^*$ from K attains an absolute minimum if and only if $\underset{\sim}{\dot{g}} - \underset{\sim}{\dot{m}}^*$ is equal to the actual rate $\underset{\sim}{\dot{r}}$ of generalized residual stress field.

The above minimum principles are equivalent to the orthogonal projections of the field $\underset{\sim}{\dot{g}}$ onto subspaces S and K, respectively.

7. NUMERICAL APPROACH

7.1 Finite element idealization

Let us apply the minimum principle derived in Section 5.5 to obtain approximated solution of the elastic-viscoplastic initial-boundary value problem. It is convenient to construct the approximate solution in the form

$$\underset{\sim}{t}(\underset{\sim}{x},t) = \sum_{k=1}^{n} \theta_k(t)\ \underset{\sim}{t}_k(\underset{\sim}{x}) \qquad (7.1)$$

resulting from the finite element idealization of the space-time region $\underset{\sim}{V}$.

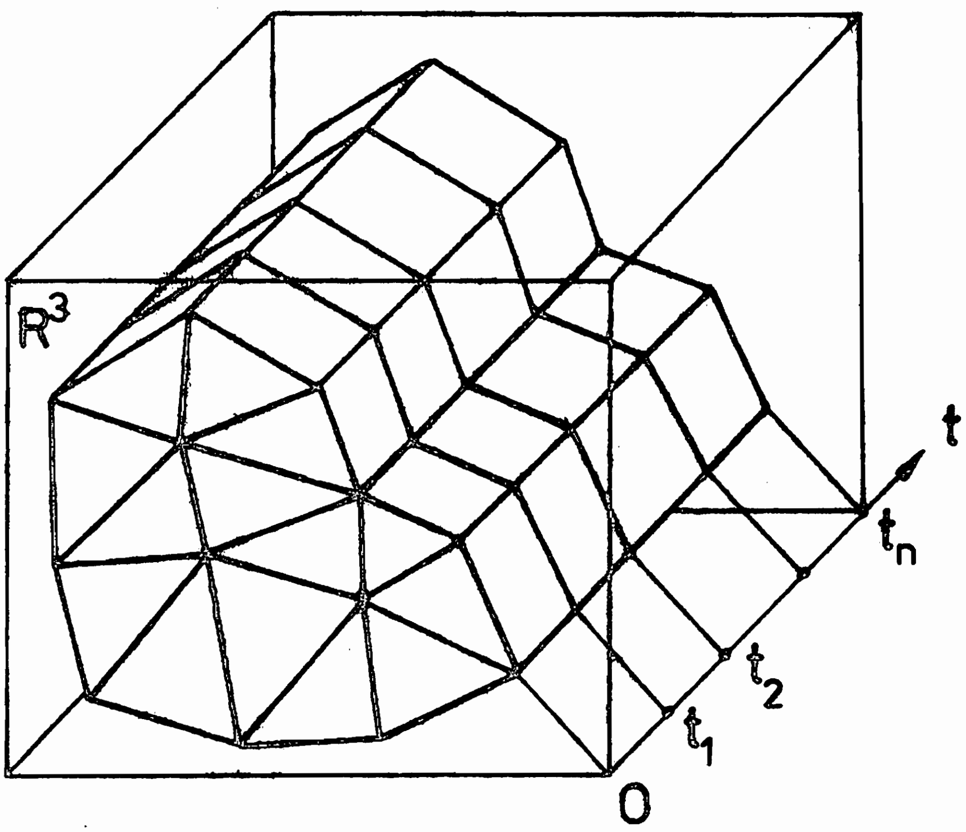

Fig. 19. Space-time finite element idealization of the region $\underset{\sim}{V}$

Here $\underset{\sim}{t}_k(\underset{\sim}{x})$ represents the generalized stress field at the moment t_k. Sectionally linear function $\Theta_k(t)$, defined in the time interval $[0,t_n]$, is presented in Fig. 20. The form (7.1) expresses the assumption of sectionally linear distribution of stress field with respect to time.

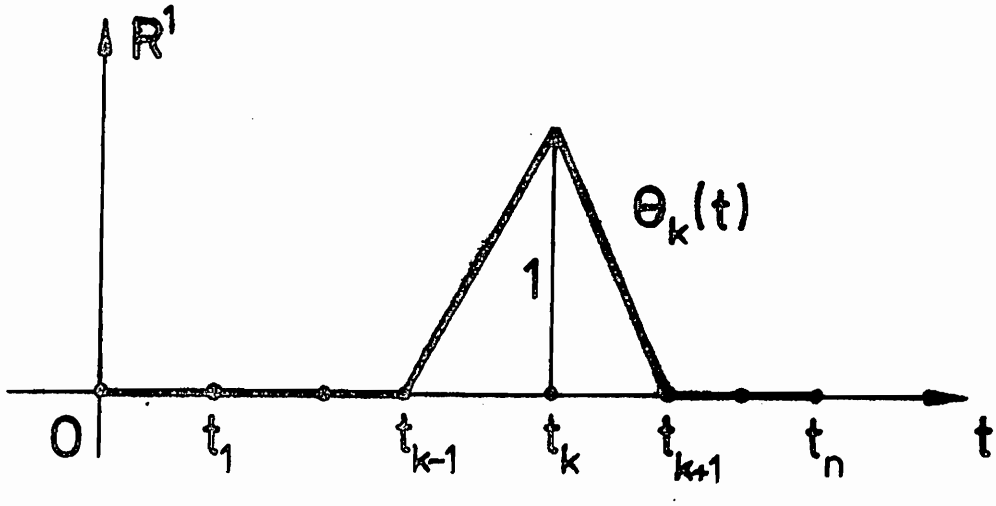

Fig. 20. Sectionally linear time function

The perfectly elastic solution $\underset{\sim}{s}^o(\underset{\sim}{x},t)$ is represented by the set of fields $\underset{\sim}{s}_k^o(\underset{\sim}{x})$, $k = 1,2,\ldots,n$, which correspond to the loadings at time t_1, $t_2,\ldots,t_n$, respectively

$$\underset{\sim}{s}^o(\underset{\sim}{x},t) = \sum_{k=1}^{n} \Theta_k(t)\ \underset{\sim}{s}_k^o(\underset{\sim}{x}) \tag{7.2}$$

Here we assume that the fields $s_k^o(\underset{\sim}{x})$, $k = 1,2,\ldots,n$ have been already obtained by the standard finite element technique for linear, elastic boundary value problems at times t_1, $t_2,\ldots,t_n$.

Consequently the statically admissible field $\underset{\sim}{r}(\underset{\sim}{x},t)$ will be presented in the form

$$\underset{\sim}{r}(\underset{\sim}{x},t) = \sum_{k=1}^{n} \Theta_k(t)\ \ \underset{\sim}{r}_k(\underset{\sim}{x}) \tag{7.3}$$

7.2 Construction of statically admissible field

To construct the statically admissible field $\underset{\sim}{r}_k(\underset{\sim}{x})$ we start from arbitrary system of forces applied to an element of the structure at its nodal points, which assures the equilibrium of the element. The set of such forces corresponding to all elements of the structure will be called the primary vector of free forces. Then we reduce the number of free forces by introducing the system of linear algebraic equations representing the force equilibrium in every node in the interior of the region V and on the boundary B_s. Eventually we obtain m-dimensional vector of free forces which determines arbitrary statically admissible field $\underset{\sim}{r}_k(\underset{\sim}{x})$ in the region V.

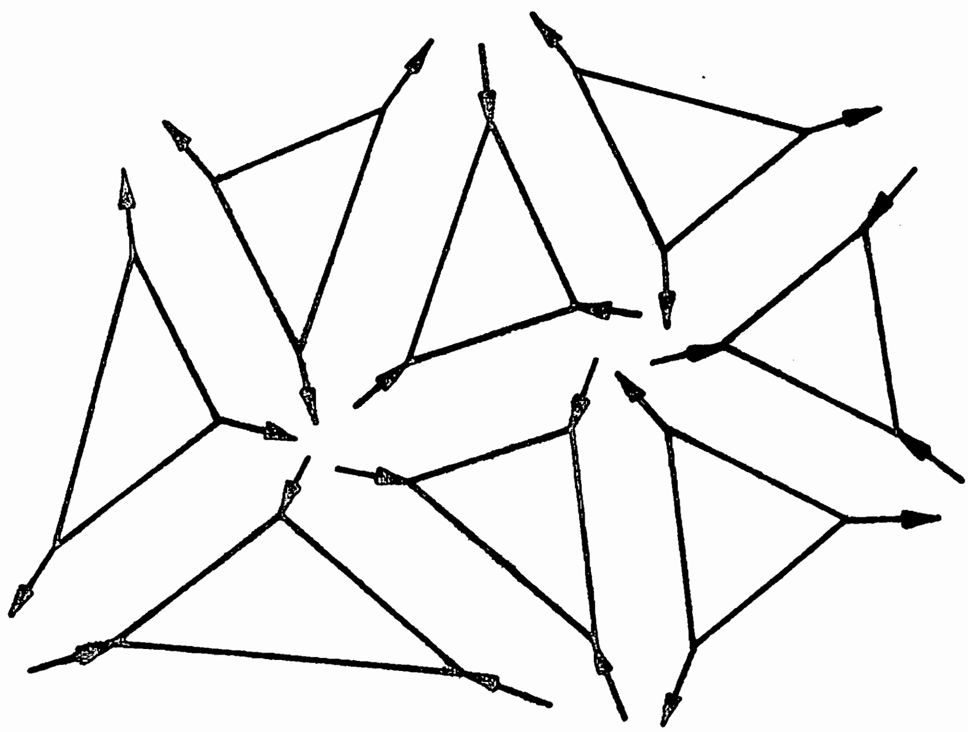

Fig. 21. Primary vector of free forces

According to the formula (7.3) the statically admissible field $\underset{\sim}{r}(\underset{\sim}{x},t)$ is represented by $(m \times n)$-dimensional matrix of free forces, which will be denoted by $\underset{\sim}{X} = X_{lk}$, $l = 1,2...,m$, $k = 1,2,...,n$. The k-th column of the matrix $\underset{\sim}{X}$ represents the state of stress at the moment t_k.

Let the function $h(t)$ appearing in the definition (5.38) of the scalar product assume the form $h_2(t)$, described in Section 5.7, where $t_o = t_n$, and let us denote the product $m \times n$ by N.

Now the set of all N-dimensional matrices $\underset{\sim}{X}$ determines the subspace $\underset{\sim}{S}'_N$, contained in the subspace $\underset{\sim}{S}'$ of all statically admissible fields.

7.3 Numerical technique

Substituting the forms (7.2) and (7.3) into (5.28) we can approximate the polar functional Φ^*_o by

$$\Phi^*_{oN}(\underset{\sim}{X}) = \sup_{r^* \in \underset{\sim}{S}'_N} [< \underset{\sim}{s}^o - \underset{\sim}{r}^*, \dot{\underset{\sim}{r}}(\underset{\sim}{X}) >_G - \Phi_o(\underset{\sim}{s}^o - \underset{\sim}{r}^*)] \tag{7.4}$$

Since we are restricted to the subspace $\underset{\sim}{S}'_N$ then $\Phi^*_{oN} \leq \Phi^*_o$ for every $\underset{\sim}{r} \in \underset{\sim}{S}'_N$.

The initial-boundary value problem is now reduced to the minimization of the functional

$$\Lambda_{oN}(\underset{\sim}{X}) = \Phi_{oN}(\underset{\sim}{X}) - < \underset{\sim}{s}^o - \underset{\sim}{r}(\underset{\sim}{X}), \dot{\underset{\sim}{r}}(\underset{\sim}{X}) >_G + \Phi^*_{oN}(\underset{\sim}{X}) \tag{7.5}$$

in the subspace $\underset{\sim}{S}'_N$. Here $\Phi_{oN}(\underset{\sim}{X}) = \Phi_o(\underset{\sim}{s}^o - \underset{\sim}{r}(\underset{\sim}{X}))$. It follows from the evaluation of the polar functional that $\Lambda_{oN} \leq \Lambda_o$ in the subspace $\underset{\sim}{S}'_N$.

Geometrical interpretation of the numerical minimization is given in Fig. 23. The horizontal plane represents the subspace $\underset{\sim}{S}'$ of all statically admissible fields. Our numerical minimization is restricted to the subspace $\underset{\sim}{S}'_N$, represented by the horizontal axis. The approximated solution is represented by vector $\underset{\sim}{r}_{(N)}$.

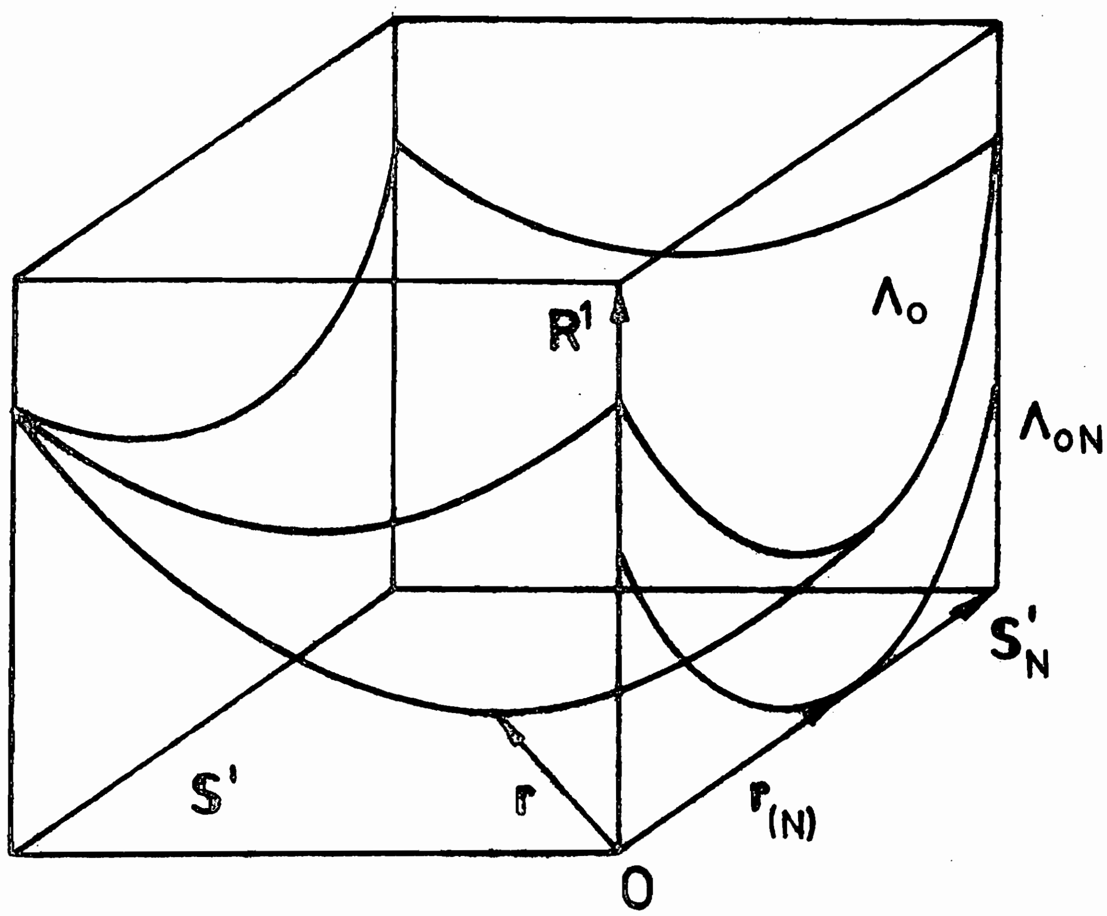

Fig. 23. Numerical minimization in the subspace $\underset{\sim}{S}'_N$.

7.4 Relation to the implicit technique

Let us consider the particular case of the <u>elastic plastic material</u>. Then we can reduce our numerical problem to minimization of the functional

$$\Lambda_{oN}(\underset{\sim}{X}) = \sup_{r^* \in \underset{\sim}{s}^o - \underset{\sim}{E}^o_N} \langle \underset{\sim}{r}(\underset{\sim}{X}) - \underset{\sim}{r}^*, \dot{\underset{\sim}{r}}(\underset{\sim}{X}) \rangle_G \qquad (7.6)$$

in the region $\underset{\sim}{s}^o - \underset{\sim}{E}^o_N$, where $\underset{\sim}{E}^o_N$ is the intersection of the plane $\underset{\sim}{s}^o + \underset{\sim}{S}'_N$ and the elastic region $\underset{\sim}{E}$.

In the particular case of time idealization, where the set of time nodes $t_1, t_2, \ldots, t_n$ is reduced to one node t_1, the stress rate $\dot{\underset{\sim}{r}}$ can be expressed in the form

$$\dot{\underset{\sim}{r}}(\underset{\sim}{X}) = \underset{\sim}{r}(\underset{\sim}{X}) \;/\; t \tag{7.7}$$

Substituting the above relation into (7.6) we conclude that the functional Λ_{ON} attains an absolute minimum if and only if the distance $|||\underset{\sim}{r}|||_G$ from the origin of the subspace $\underset{\sim}{S}'_N$ to the convex set $\underset{\sim}{s}^o - \underset{\sim}{E}^o_N$ attains an absolute minimum. The last statement coincides with the notion of the implicit technique proposed by MOREAU in [7].

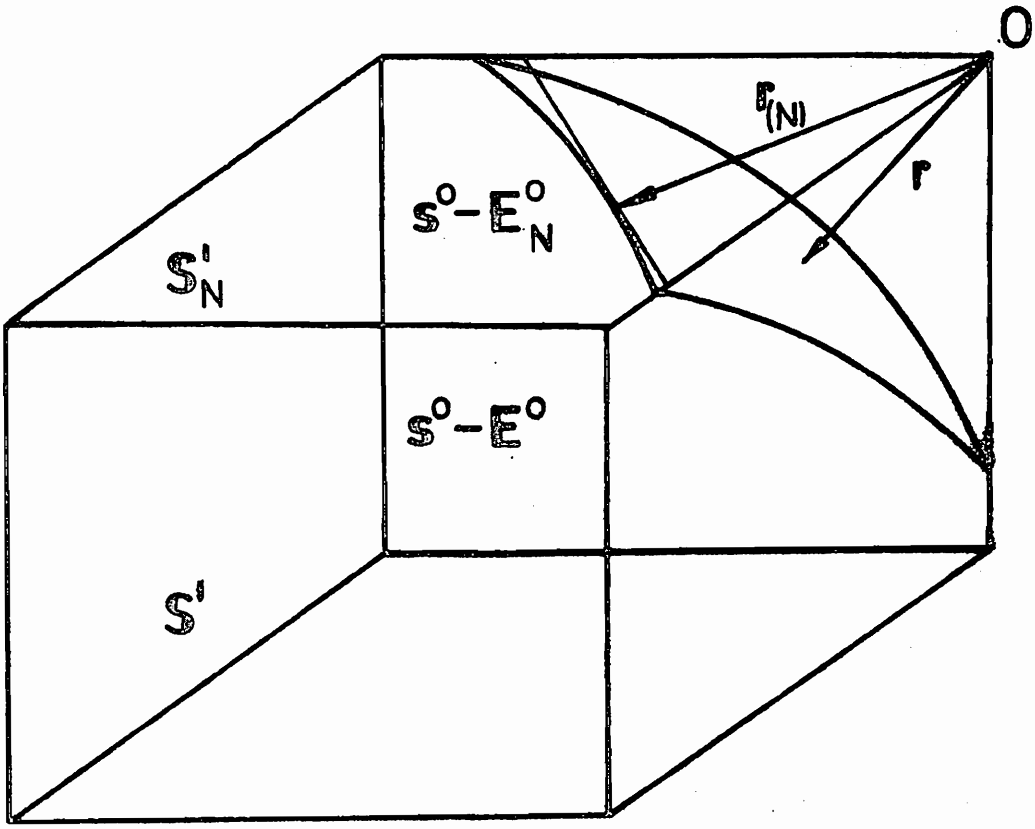

Fig. 24. Implicit technique in the subspace $\underset{\sim}{S}'_N$.

It should be noted that in the considered case of one time step we can also construct the scalar product using the function $h_1(t)$ described in Section 5.7. Since the value $\underset{\sim}{r}_1(\underset{\sim}{x})$ at time t_1 uniquely determines the history of $\underset{\sim}{r}(\underset{\sim}{x},t)$ then the form $h_1(t)$ assures that the scalar product $<\underset{\sim}{r},\dot{\underset{\sim}{r}}>_G$ is strictly convex.

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