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Uwe Kohlberg

Variational Principles and theirNumerical Application to Geometrically Nonlinear v. Karman Plates

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Referents:

Prof. Dr.-Ing. H. Stumpf
Prof. Dr.-Ing. K.-H. Schrader

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SUMMARY

Following well-known variational principles of the linear plate theory a variational formulation with relaxed continuity requirements of the von Kármán-plate theory will be established, which is suitable for numerical applications.
Proceeding from the basic principle of the stationary value of potential energy the Legendre transformation in dual variables produces the Hellinger-Reissner functional and in addition a functional that is defined as a Hu-Washizu functional with respect to the membrane part of the von Kármán-plate. The formulation of the Hu-Washizu functional is found by the analogy between the problems of stretching and bending of plates.
Removing all side conditions the functionals are presented in their generalized and modified form.
Thus, by these general representations the application of various finite-element shape functions is allowed.
The numerical part of the present work is referring only to linear or constant local limited shape functions of Ritz type. Subsequently, the optimization of the functionals results in solving nonlinear equations by the iterative scheme of the Newton-Raphson method.

ZUSAMMENFASSUNG

Ausgehend von bekannten Variationsprinzipien der linearen Plattentheorie findet eine für die numerische Anwendung geeigente Variationsformulierung mit gelockerten Stetigkeitsbedingungen für die von Kármán'sche Plattentheorie statt.
Unter Zugrundelgung des Prinzips vom stationären Wert des Gesamtpotentials führt die Legendre Transformation in den jeweisl zueinander dualen Feldgrößen auf das Hellinger-Reissner Funktional und ein bezüglich des Membrananteils der Kármán-Platte definiertes Hu-Washizu Funktional. Die Formulierung des Hu-Washizu Funktionals geht auf die zwischen Scheibe und Platte bestehende statisch-geometrische Analogie zurück.
Die Befreiung aller Nebenbedingungen stellt beide Funktionale in ihrer verallgemeinerten und modifizierten Fassung vor.
Die auf diese Weise allgemein gehaltenen Darstellungen gestatten so die Anwendung der verschiedenartigsten Finite-Element-Ansätze. Im numerischen Teil der vorliegenden Arbeit werden ausschließlich lineare oder konstante

lokal begrenzte Ritz-Ansätze benutzt. Die Optimierung der Funktionale geschieht anschließend durch iterative Lösung nichtlinearer Gleichungssysteme mit Hilfe des Newton-Raphson-Verfahrens.

NOTATIONS

Coordinate systems

x^i	Fixed Cartesian coordinate system
Θ^α	Material coordinate system

Signs of operations

$i,j,\ldots = 1,2,3$	Indices of the threedimensional Euclidean space
$\alpha,\beta,\ldots = 1,2$	Indices of the twodimensional Riemannean space
$(\ldots)_{,\alpha}$	Partial derivative with respect to Θ_α
$(\ldots)\vert_\alpha$	Covariant derivative with respect to Θ_α
C	Exterior boundary
$C_1,\ C_2$	Exterior boundaries of geometrical and statical field quantities
Γ	Interelement boundary
$(\ldots)^*$	Prescribed field quantities
$(\ldots)_+, (\ldots)_-$	Reference to positive and negative interelement boundaries
$\langle\ldots\rangle$	Discontinuity
$\circ$	Symbol for any product of two tensors
$\underset{\sim}{\nabla}$	Nabla-Operator
$\underset{\sim}{I}$	Unit tensor

Geometrical quantities

$\underset{\sim}{r}$	Vector of position
$\underset{\sim}{g}_i;\ a_\alpha,\ \underset{\sim}{n}$	Base vectors of the space; of a surface
$\underset{\sim}{t},\ \underset{\sim}{\nu},\ \underset{\sim}{n}$	Triad of a curve on an surface
$a_{\alpha\beta}$	Metric tensor
$b_{\alpha\beta}$	Tensor of change of curvature
$\Gamma^\alpha_{\beta\gamma}$	Christoffel symbol
$\epsilon_{\alpha\beta}$	Permutationtensor

$R_{\alpha\beta\lambda\mu}$	Riemann-Christoffel tensor
κ_t, κ_ν	Geodetic curvatures of a curve
$b_{\nu t}$; b_{tt}; $b_{\nu\nu}$	Torsion; normal curvatures of a surface

Statical and kinematical quantities

$N^{\alpha\beta}$, $M^{\alpha\beta}$	Tensor of membrane stresses, of bending moments
$\gamma_{\alpha\beta}$, $\kappa_{\alpha\beta}$	Strain tensors
u_α, u_3	Tangential and normal deflection
F	Airy's stress function
$\Theta_{\alpha\beta}$	Linear strain tensor
$\underset{\sim}{\beta}$	Rotation vector of the normal $\underset{\sim}{n}$ on a surface
β_ν	$= - u_{3,\nu}$
β_t	$= - u_{3,t}$
$\underset{\sim}{\chi}$	Force vector of membrane stresses
χ_ν	$= F_{,t}$
χ_t	$= -F_{,}$
$P^*_{t\nu}$, $P^*_{\nu\nu}$	Prescribed membrane forces on C_2
K^*_ν	Prescribed bending moment on C_2 and Γ
P^*	Vertical loads per unit area
P^*_n	Vertical loads per unit length of C_2 and
K^*_t	Singular vertical force on any corner of C_2
D^*_t	Prescribed stretching on C_1 and Γ
k^*_{nt}	Prescribed effective change of curvature on C_1 and Γ
D^*_ν	Prescribed singular shear strain at any corner of C_1

Quantities of energy

W	$= W(\gamma_{\alpha\beta},\kappa_{\alpha\beta})$ function of strain energy
W_C	$= W_C(N^{\alpha\beta},M^{\alpha\beta})$ function of complementary energy
W_M	$= W_M(\kappa_{\alpha\beta})$ strain energy of stretching

W_B	$= W_B(\kappa_{\alpha\beta})$	strain energy of bending
W_{CM}	$= W_{CM}(N^{\alpha\beta})$	complementary energy of stretching
W_{CB}	$= W_{CB}(M^{\alpha\beta})$	complementary energy of bending

<u>Quantities of material</u>

$H^{\alpha\beta\lambda\mu}$	Tensor of elastic constants
$E^{\alpha\beta\lambda\mu}$	Inverse tensor of elastic constants
E	Young's modulus
ν	Poisson's ratio

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Appendix A

Appendix B

Appendix C

INTRODUCTION

In geometrically linear plate theory the finite-element-method has found a wide range of application. In case of Kirchhoff-type plate theory there is a boundary value problem of fourth order by the bipotential equation in the plate deflection u_3 with geometric boundary conditions of first order. The equivalent variational principle contains under the surface integral derivatives of second order in u_3. The boundary integrals are described by u_3 itself and by the normal derivative $u_{3,\nu}$. This means, that after finite transformation the applied shape functions have to be twice differentiable and continuity up to the first derivative has to be assured at interelement boundaries. Rectangular conforming plate elements are easily given by Schäfer [1] by means of Hermite interpolation polymonials. From the mathematical point of view it suffices, to choose twelve modal parameters for each element, in order to satisfy the continuity requirements mentioned above. However, the finite-element-method requires sixteen for the description of a constant state of twisting. Conforming triangular elements exhibit already numbers of parameters up to twenty-one.
Starting from the principle of the minimum of potential energy, the numerical treatment of rectengular elements by the aid of Hermite - polynomials may be seen e. g. in [3] and [4]. Linear trial functions for stress functions in linear plate bending theory constitute the basis for numerical applications of the complementary functional. Numerical calculations, shear forces including and neglecting, are carried out by Knothe [5] after Hoppe [6] had worked before with quadratic shape functions.
Proceeding from the principle of Hellinger-Reissner, mixed finite-element-models of plate bending are presented in the numerical preparation of Connor [7]. He takes into consideration the influence of shear forces too. These functionals are fomulated - without shear deformation in *reduced form* - by involving jump terms. Thereby, without reference to the theory of distribution, the jump conditions are developed by means of a simple representation of sequences for the Heaviside function and the delta function as well as the doublet function as its both derivatives. The advantage of this procedure, pointed out by Connor, exists in a formal way of application, but it is disadvantageous, because direct imagination has been lost. Another detailed explanation of the modified Hellinger-Reissner functional (attached with jump terms) of linear plate theory represents Washizu [8] by the aid of Lagrange multipliers.

Mixed finite-element-procedures on the basis of stress functions by Schäfer in connection with the tensor of change of curvature are not performed up to now in the literature. Likewise, there is no mixed model developed by the strain tensor in linear membrane theory.
Originially, the desire to remove high continuity requirements in variational functionals was initiated by some considerations of Pian [9], [10]. First of all, it is a question of modification of the complementary energy functional by means of trial functions, which satisfy the equilibrium equations. These functions are polynomials, and they do not exist of nodal but of so-called generalized parameters. Thereby, the unsatisfied transitional conditions are taken inot consideration by means of continuous shape functions with nodal parameters. A subsequent elemination process takes care that the underlying functional contains only displacement parameters. The corresponding geometric model starts from the modified functional with displacements as independent variables and makes use of polynomials for displacement fields. By Pian both formulations are called hybrid models and they were applied at first to membranes. A hybrid performance for Reissner's functional is given by Pian in [11]. Referring to this functional Connor [7] presents numerical calculations for plate bending theory. Examimations of Prager [12] are also based on the Reissner functional of linear plate theory. Discontinuities of static field variables are allowed, whereas dual geometric quantities must be contninuous.
The *reduced form* of the functionals mentioned above arises from a simple reflection by Herrmann [13], [14]. The decisive idea has to be seen in integration by parts, so that to a certain extent continuity requirements of a field variable can be shifted off to the dual. Thus in case of plate theory, the bending moment and the deflection only must be continuous. On the contrary, the effective shear force and the first derivatives of the deflection may be discontinuous. These facts are included in those cases shown by Connor [7]. Sometimes, the literatur refers to this functional explicitly as the *Herrmann - functional.*
Bufler [16], [17] has given up the restrictions made by Prager. All variables need not to be continuous. The variational principle is formulated in such way, that on interelement boundaries the jumps of static field variables correspond to mean values of the dual geometric variables respectively and vice versa. Continuity prevails in nodal points only. Numerical calculations do not exist. Prager as well as Bufler do not represent their functionals in a reduced form.

A detailed summary of the variational principles considered above for linear plate theory is found in the contribution of Knothe [18].

The first calculations of geometrically nonlinear plate problems proceed from differential equations of the von Kármán plate. So, Levy [19] has treated various types of suppoerted plates with different loads by choosing global shape functions for the deflection and the Airy stress function; Poisson's ratio was chosen to $\nu = 0{,}316$. Way [20] makes use of the principle of stationary value of potential energy, in order to examine the clamped plate by application of global trial functions in displacements. An extensive discussion about von Kármán's equations and pertaining variational functionals offers Wolmir [21] in his standard book. Therein, iterative schemes of numerical calculations are demonstrated too, which start from Euler's equation and from the energy integral respectively. Moreover, instability problems are discussed. Bergan and Clough [22] apply a conforming quadrilateral element as displacement model to the variational functional in displacements of the von Kármán-plate. The selected model consists of four triangles with nineteen degrees of freedom altogether. The solution results in solving the nonlinear equations by applying Newton-Raphson's process. Just so, the authors of the treatise [23] proceed from the variational functional displacements. They use a bilinear appoximation for membrane displacements and additionally bicubic Hermite-polynomicals for the deflection. The iterative solution of the system of nonlinear equations takes place by successive approximations with linear systems of equations. The conjugate variational functional for the nonlinear shallow shell theory of Donnel-Marguerre and for the von Kármán-plate constitutes the basis for finite-element formulations of Gass-Tabarrok [24]. By means of Hermite-polynomials rectangular element models with twelve parameters are derived for the deflection and the Airy stress function. The algorithm for solving nonlinear equations follows by applying the Newton-Raphson procedure and the incremental method. Further details about other possibilities of finite-element calculations for the von Kármán-plate are described by Tabarrok and Dost in [25]. It is a matter of generalizations of the conjugate functional and of that in displacements. The static-geometric analogy, indicated by the Airy stress function is not followed. Basic examinations with respect to convex properties of the functional in displacements have led to the complementary functional of the von Kármán-plate. This was done by Stumpf [26], [27], [28] using the transformation of Legendre. As pointed out, under certain conditions both functionals become appearent to be extremum principles. You will find more details for nonlinear shells in [29]. For the complementary functional of the plate numerical calculations with global shape functions are published in [30]. Up to now, there was no success in getting numerical results for it by the finite-element-scheme. Nemat-Nasser [31], [32] was the firs, who removed all continuity requirements in variational principles of geometrically nonlinear elasticity problems. An extension to geometrically nonlinear plates and shells has not been yet performed. On the other hand, use has been made already by the reduced form of generalized nonlinear variational functionals [33].

The present work starts from the contribution of Nemat-Nasser [31], [32] and relates them to the von Kármán-plate. Furthermore, the representations of Prager [12], Herrmann [13] and Bufler [16] will be put as a foundation too. The modified functional of Hellinger-Reissner and in addition a functional, that is defined in the sense of Hu-Washizu exclusively with respect to the membrane part of the von Kármán-plate, are to be discussed. In chapter 5 this functional will be derived as a new variational principle from the functional in displacements. Thereby, static-geometric analogies based on examinations of Elias [34], [35], [36] turn out to be very useful. Finally, this consideration of analogy reveals all continuity requirements to the stress function appearing in the corresponding boundary integral of the variational expression (s. chapter 5.2.). With regard to later possible reflections for shallow shells a tensor formulation is preferred and so distance will be kept from the x,y-description of Elias. Separately, in chapter 3 against common use all jump terms are not developed individually but directly general for surface tensors of any order. Subsequently, both functionals are represented in chapter 7.2 in the reduced form by permitting all discontinuities. Until now, discontinuities of field variables in nodal points have not been regarded; they will be involved in the following for the sake of clearness.
According to special local limited shape functions of Ritz-type various types of modifications are forthcoming in the treatment of numerical examples ciscussed in chapter 8.2.. Especially, constant shape functions for membrane stresses have proved to be very advantageous. Moreover, completely unknown are trial functions for the state of membrane strains related to the Airy stress function in the Hu-Washizu functional, because this functional itself is introduced as a new variational principle.
It shall be not unmentioned, that calculations with partly local and global shape functions respectively have been proved. By this means, rather good

results were obtained. One could imagine, that it might be possible to apply this concept on element level too by using shape functions of different degrees of freedom. Then, by means of the corresponding equations for interior and boundary nodal points it might be checked, if possibly the system of equations became ill-conditioned.
In view of tensorial formulations in the present work most of the important formulae of tensor calculus are prepared separately in chapter A1 to A3. This representation is self-contained and for general considerations it is referred to surfaces of any curvature. In special case of a plane, regarding von Kármán-plates, the tensor of curvature equals zero. The notation of tensor descriptions chosen here is similar to that from Pietraszkiewicz [37]. Particular attention will be payed to the facts arising on the boundary of the plate, in order to guarantee complete formulations at curved edges. This desire will be supported by the treatise of chapter B, which refers to surface edges of any curvature. Initial starting-points to that problem for plane surfaces are disclosed in a publication of Fraeijs de Veubeke and Zienkiewicz [38] but without any detailed prove.
Appendix C1 gives a glimpse into the formulation of generalized integral theorems. The proof of these theorems can be looked up in fundamental books like those of Gurtin [39], Lagally [40], Kästner [41] or Trostel [42]. Special integral transformations generated by the cited integral theorems are made available in chapter C2 for the descriptions of the chapter 1 to 9. Chapter 10 gives a graphical demonstration of numerical results.
After getting the system of nonlinear equations by carrying out the first variation, all functionals were optimized by the iterative scheme of Newton-Raphson. In doing so, it was aspired to optimize the functionals, which were determined for numerical calculation, only by linear or constant element shape functions.
Thereby, it was not necessary to write a new program for the Newton-Raphson procedure, because this could be borrowed from the computer library of the Ruhr-Universität Bochum. An incremental formulation with numerical results can be inspected in chapter 9 and 10.
In order to compare the results of the treated variational principles, all examples were referred to the simply supported plate with unmovable edges.
In addition, the clamped plate with movable edges will be considered. Furthermore, an example, where the plate is loaded by a singular force, will be of interest too.

1. INTRODUCTURY REMARKS TO THE EULEREAN AND LAGRANGEAN DESCRIPTION

In geometrically linear theory the equilibrium statements are formulated by means of descriptions with respect to the undeformed body. This is justified only for small deformations. However, if the elastic body suffers large deformations, then the state of equilibrium depends on displacements. That is why equilibrium conditions must be established in the deformed state of the system, and this is done in differential form as well as in linear theory.

Subsequently, the connection of the displacement relations with interior stresses by Hooke's law offers the possibility, to ascertain strains and stresses of the body. Formally, there is no difference between geometrically linear and nonlinear theory, only the descriptions of both theories are quite different. In the first, it is necessary to perform all descriptions within the undeformed system, the second however makes it possible to describe all facts within the deformed or the undeformed (reference) state.

Mathematically, this is done by introduction of two coordinate systems x^i and X^i of the undeformed and the deformed state. Then, the behavior of the deformation can be seized analytically by

$$x^i = x^i(X^j) \tag{1.1}$$

or

$$X^i = X^i(x^j) \quad , \quad (i,j = 1,2,3) \tag{1.2}$$

If both functions are uniquely defined and continuous with continuously partial derivatives, they are inverse formulations one of each other. This is true, only if the *Jacobian determinant* J is neither zero nor infinite.

$$J = \det\left(\frac{\partial X^i}{\partial x^j}\right) \tag{1.3}$$

The coordinates x^i alone fix a point solely in the cartesian frame $\underset{\sim}{i}_i$. They do not suffice in general, to say something about geometries of the undeformed body. To do so, curvilinear coordinates Θ^j $(j = 1,2,3)$ are introduced.

The coordinates

$$x^i = x^i(\Theta^j) \tag{1.4}$$

are known as *Lagrangean* or *material coordinates*.
The state of deformation is completely defined by Lagrange-coordinates, if (1.4) is substituted into (1.2). This kind of description is called *Lagrangean* or *material approach.*

$$X^i = X^i(\Theta^j) \tag{1.5}$$

Analogous regards with respect to the deformed state of the body lead to the representation

$$X^i = X^i(\bar{\Theta}^j) \tag{1.6}$$

where the new introduced coordinates $\bar{\Theta}^j$ measure the deformed body. Coordinates, indicated by (1.6), are called *Eulerian* or *local coordinates* as well. In order to develop *Eulerian* or *local descriptions* of the boundary problem the relations of (1.6) are substituted into those of (1.1). This yields

$$x^i = x^i(\bar{\Theta}^j) \tag{1.7}$$

In practice, both descriptions are performed by the displacement vector $\underset{\sim}{u}$.
For instance, in Lagrangean description the displacement vector is referred to the known basis $\underset{\sim}{g}_i$ of the undeformed system.

$$\underset{\sim}{u} = u^i \underset{\sim}{g}_i \tag{1.8}$$

Since the base vectors depend on Θ^j, this is also valid for the components of $\underset{\sim}{u}$.

$$u^i = u^i(\Theta^j) \tag{1.9}$$

By this, relation (1.5) is received again, because the position vectors $\bar{\underset{\sim}{r}}$, $\underset{\sim}{r}$ of the deformed and the undeformed system are related to each other by

$$\bar{\underset{\sim}{r}} = \underset{\sim}{r} + \underset{\sim}{u} \,. \tag{1.10}$$

This means in components by the aid of (1.4)

$$X^i = x^i(\Theta^j) + u^i(\Theta^j) = X^i(\Theta^j) \tag{1.11}$$

Completely analogous are the facts for the Eulerian description. The displacement vector is represented in the known basis $\bar{\underset{\sim}{g}}_i$ of the deformed system.

$$\underset{\sim}{u} = \bar{u}^i \underset{\sim}{g}_i \tag{1.12}$$

Thus, as explained above, the components are concluded to be dependent of Eulerian parameter lines $\bar{\theta}^j$.

$$\bar{u}^i = \bar{u}^i(\theta^j) \tag{1.13}$$

Now, it follows from (1.10)

$$\underset{\sim}{r} = \bar{\underset{\sim}{r}} - \underset{\sim}{u}$$

or

$$x^i = x^{-i}(\theta^j) - \bar{u}^i(\bar{\theta}^j) = x^i(\bar{\theta}^j) \tag{1.14}$$

and that is identical to (1.7).

Eulerian's description is unsuitable for practical calculation, since the base vectors of the deformed body are not known. They are present not before the state of displacement is ascertained. This is the reason, why the Lagrangean approach must be applied.

2. THE PRINCIPLE OF THE STATIONARY VALUE OF POTENTIAL ENERGY

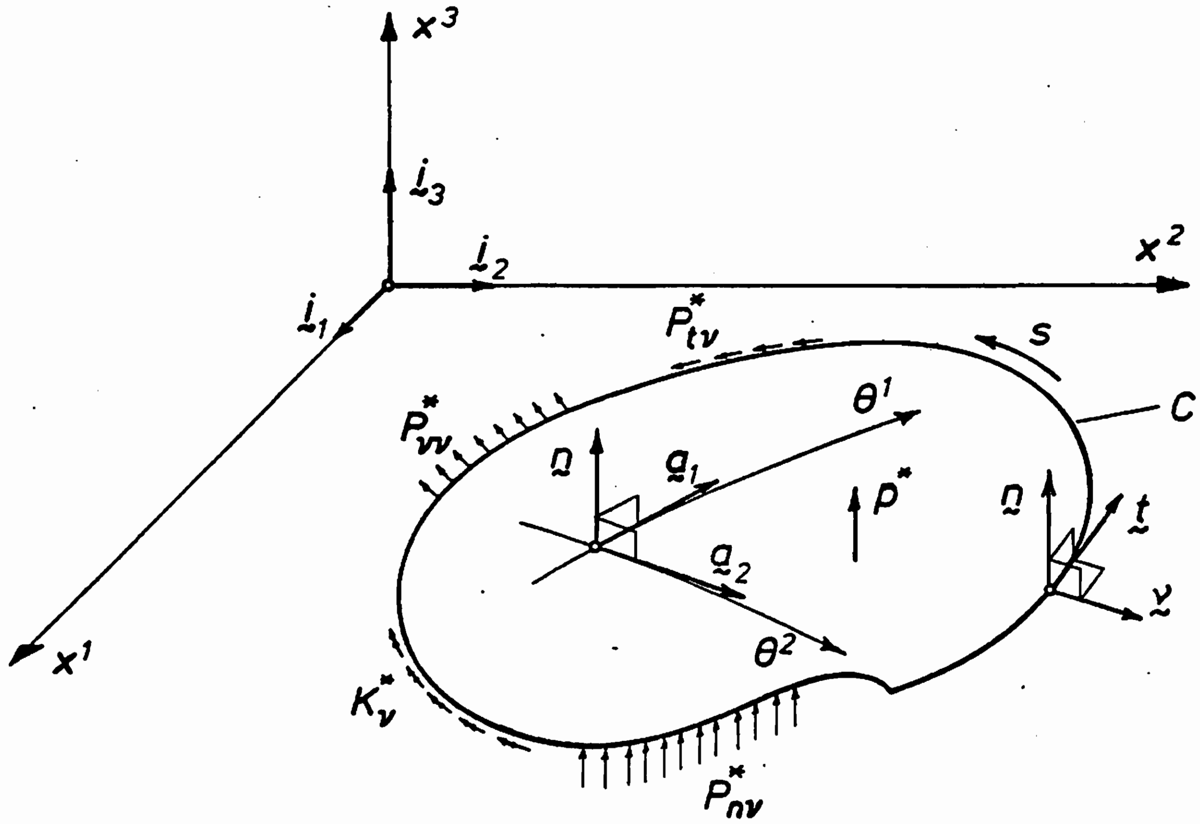

Fig. 1

Starting from threedimensional elasticity theory, the governing equations of the boundary value problem are reduced to the basis $\underset{\sim}{a}_\alpha$ $(\alpha = 1,2)$ of the middle surface in thin shell theory by introducing appropriate field variables. Thereby, the middle surface is defined in such way, that each of its points bisects the thickness h of the shell-structure.
Taking into account the deformations and their relation to statical field variables, additional assumptions allow to develop various types of consistent shell theories. In case of the von *Kármán-plate theory* the following assumptions exist:
The plate is homogeneous, elastic and isotropic and the state of stresses is approximately plane and parallel to the middle surface. Strains are small everywhere. Within *moderate rotation* theory, as it holds for the von Kármán-plate, squares of the rotations around tangents to the middle surface are small of the same order as the strains, whereas rotations around normals perpendicular to the middle surface are neglegibly small. Hooke's law is valid and the hypothesis of Kirchhoff-Love may be assumed [43].

Within the range of these assumptions the deformation of the plate continuum is described by two *surface strain measures*, which are referred to the basis $\underset{\sim}{a}^{\alpha}$ (α = 1,2) of the plate middle surface:

$$\gamma_{\alpha\beta} = \frac{1}{2}(\bar{a}_{\alpha\beta} - a_{\alpha\beta}) \tag{2.1}$$

$$\kappa_{\alpha\beta} = -(\bar{b}_{\alpha\beta} - b_{\alpha\beta}) \tag{2.2}$$

The quantities $\gamma_{\alpha\beta}$ are known as components of the *surface Lagrangean strain tensor* while $\kappa_{\alpha\beta}$ are called the components of the *Lagrangean tensor of change of curvature* of the surface.

Geometric quantities, supplied with a dash, are related to the deformed surface, and they are expressed in terms of geometric quantities of the reference surface, because *Lagrangean approach* is used.

Since $b_{\alpha\beta}$ = 0 for the von Kårmån-plate the components of the strain measures turn out to be

$$\gamma_{\alpha\beta} = \theta_{\alpha\beta} + \frac{1}{2}\varphi_{\alpha}\varphi_{\beta} \tag{2.3}$$

$$\kappa_{\alpha\beta} = -\varphi_{\alpha}/_{\beta}\ . \tag{2.4}$$

Therein, $\Theta_{\alpha\beta}$ indicates the linear strain tensor.

$$\theta_{\alpha\beta} = \frac{1}{2}(u_{\alpha}/_{\beta} + u_{\beta}/_{\alpha}) \tag{2.5}$$

The componenets φ_{α} are known as *linearized rotations of the normal* to the surface [37].

$$\varphi_{\alpha} = u_{3,\alpha} \tag{2.6}$$

By means of the principle of virtual work one gets the principle of the stationary value of potential energy in Lagrangean description. This has to be done on the premises of conservative forces by integration through the thickness of the plate.

$$\begin{aligned} I = & \int_A W(\gamma_{\alpha\beta}, \kappa_{\alpha\beta})\, dA & & - \int_A p^* u_3\, dA \\ & - \int_{C_2} (P^*_{\nu\nu} u_\nu + P^*_{t\nu} u_t)\, ds & & - \int_{C_2} P^*_{n\nu} u_3\, ds \\ & - \int_{C_2} K^*_\nu \beta_\nu\, ds & & - [\, K^*_t u_3 \,]_{C_2} \end{aligned} \tag{2.7}$$

p^* and $P^*_{n\nu}$ stand for vertical loads in the interior and on the boundary of the plate. $P^*_{\nu\nu}$ and $P^*_{t\nu}$ represent prescribed membrane forces and K^*_ν signifies a predicted bending moment, while by K^*_t singular forces are taken into account, which are acting upon single corner of the edge.

$$[K_t^* u_3]_{C_2} = \sum_{1}^{N_2} [M_{t\nu}^*(s_i+0) - M_{t\nu}^*(s_i-0)]\, u_3(s_i)\, |_{C_2} \tag{2.8}$$

All prescribed quantities are measured per unit area and unit length respectively of the undeformed configuration. C_1 signifies that part of the whole boundary, where geometric field quantities are prescribed. On the other segment C_2 static boundary conditions exist. Out of

$$\left.\begin{aligned} \gamma_{\alpha\beta} - \tfrac{1}{2}(u_\alpha|_\beta + u_\beta|_\alpha + u_{3,\alpha}u_{3,\beta}) &= 0 \\ \varkappa_{\alpha\beta} + u_3|_{\alpha\beta} &= 0 \end{aligned}\right\} \tag{2.9}$$

from (2.3), (2.4) and (2.6) the functional is related to further subsidiary conditions on C_1:

$$\left.\begin{aligned} u_\nu - u_\nu^* &= 0 \qquad & u_3 - u_3^* &= 0 \\ u_t - u_t^* &= 0 \qquad & \beta_\nu - \beta_\nu^* &= 0 \end{aligned}\right\} \tag{2.10}$$

Within *moderate rotation* plate-theory the relation $\varphi_\nu = -\beta_\nu$ holds on the boundary, where the vector

$$\underset{\sim}{\beta} = \bar{\underset{\sim}{n}} - \underset{\sim}{n} = \beta_\alpha \underset{\sim}{a}^\alpha$$

measures rotation of the normal $\underset{\sim}{n}$ [37]. Deriving the functional (2.7), use has been made already of linear elastic behavior of the material based on Hooke's law.

The *function of energy density* $W(\gamma_{\alpha\beta}, \kappa_{\alpha\beta})$ as well as the *function of complementary energy density* $W_c(N^{\alpha\beta}, M^{\alpha\beta})$ are quadratic forms of their independent variables and they are positive definite.

$$\left.\begin{aligned} W(\gamma_{\alpha\beta}, \varkappa_{\alpha\beta}) &= \tfrac{1}{2} H^{\alpha\beta\lambda\mu}\left(\gamma_{\alpha\beta}\gamma_{\lambda\mu} + \tfrac{h^2}{12}\varkappa_{\alpha\beta}\varkappa_{\lambda\mu}\right) \\ &= W_M(\gamma_{\alpha\beta}) + W_B(\varkappa_{\alpha\beta}) \\ W_c(N^{\alpha\beta}, M^{\alpha\beta}) &= \tfrac{1}{2} E_{\alpha\beta\lambda\mu}\left(N^{\alpha\beta}N^{\lambda\mu} + \tfrac{12}{h^2} M^{\alpha\beta}M^{\lambda\mu}\right) \\ &= W_{cM}(N^{\alpha\beta}) + W_{cB}(M^{\alpha\beta}) \end{aligned}\right\} \tag{2.11}$$

Herein, the tensor $H^{\alpha\beta\lambda\mu}$ of elastic constants and $E_{\alpha\beta\lambda\mu}$ as its inverse have the following form:

$$\left.\begin{aligned} H^{\alpha\beta\lambda\mu} &= \frac{Eh}{2(1+\nu)}\left[a^{\alpha\lambda}a^{\beta\mu} + a^{\alpha\mu}a^{\beta\lambda} + \frac{2\nu}{1-\nu}a^{\alpha\beta}a^{\lambda\mu}\right] \\ E_{\alpha\beta\lambda\mu} &= \frac{1+\nu}{2Eh}\left[a_{\alpha\lambda}a_{\beta\mu} + a_{\alpha\mu}a_{\beta\lambda} - \frac{2\nu}{1+\nu}a_{\alpha\beta}a_{\lambda\mu}\right] \end{aligned}\right\} \quad (2.12)$$

Using both energy density functions, *constitutive equations* are derived, which are completely equivalent to the transformation of Legendre.

$$\left.\begin{aligned} \frac{\partial W_M}{\partial \gamma_{\alpha\beta}} &= N^{\alpha\beta} \\ \frac{\partial W_{CM}}{\partial N^{\alpha\beta}} &= \gamma^{\alpha\beta} \end{aligned}\right\} \iff W_M(\gamma_{\alpha\beta}) + W_{CM}(N^{\alpha\beta}) = N^{\alpha\beta}\gamma_{\alpha\beta} \quad (2.13)$$

$$\left.\begin{aligned} \frac{\partial W_B}{\partial \varkappa_{\alpha\beta}} &= M^{\alpha\beta} \\ \frac{\partial W_{CB}}{\partial M^{\alpha\beta}} &= \varkappa_{\alpha\beta} \end{aligned}\right\} \iff W_B(\varkappa_{\alpha\beta}) + W_{CB}(M^{\alpha\beta}) = M^{\alpha\beta}\varkappa_{\alpha\beta} \quad (2.14)$$

Taking into consideration the constitutive equations (2.13), (2.14) and relations (2.3) to (2.6), the first variation of (2.7) is received by the aid of both integral transformations (C2.4) and (C2.7).

$$\begin{aligned} \delta I = &\int_A \left[M^{\alpha\beta}|_{\alpha\beta} + N^{\beta}|_{\beta} + p^*\right]\delta u_3\, dA \\ &-\int_A N^{\alpha\beta}|_{\beta}\,\delta u_\alpha\, dA \\ &-\int_{C_2}(P^*_{\nu\nu} - N_{\nu\nu})\,\delta u_\nu\, ds \quad -\int_{C_2}(P^*_{t\nu} - N_{t\nu})\,\delta u_t\, ds \\ &-\int_{C_2}(K^*_{\nu} - M_{\nu\nu})\,\delta\beta_\nu\, ds \quad -\int_{C_2}\left[P^*_{n\nu} - (P_{n\nu} + N_\nu)\right]\delta u_3\, ds \\ &+\left[(K^*_t - M_{t\nu})\,\delta u_3\right]_{C_2} \end{aligned} \quad (2.15)$$

The *fundamental lemma* of variational calculus yields the *equilibrium* and *statical boundary conditions* as *natural conditions*.

The vertical components of membrane forces and the effective shear force were introduced in (2.15) by the abbreviations

$$N^{\beta} \doteq N^{\alpha\beta} u_{3,\alpha} , \qquad N_{\nu} = N^{\beta} \nu_{\beta} \tag{2.16}$$

$$P_{n\nu} = M_{\nu\nu,\nu} + 2 M_{t\nu,s} - \kappa_t (M_{tt} - M_{\nu\nu}) \tag{2.17}$$

Therein, κ_t is the geodesic curvature of the boundary curve.
Some contributions of Stumpf [26], [27], [28] are pointing out, when the functional achieves a minimum, the solution is uniquely defined and for what class of functions this is valid if occasion arises.

3. TO THE DEFINITION OF DISCONTINUITIES

The functional (2.7) reveals, that admissable shape functions must satisfy certain requirements of continuity and of differentiability. In the next chapter the functional shall be modified in such way that all requirements to the shape functions may be dropped and discontinuities are allowed. The formulation of these facts demands some reflections, which are to be explained regarding figure 2.

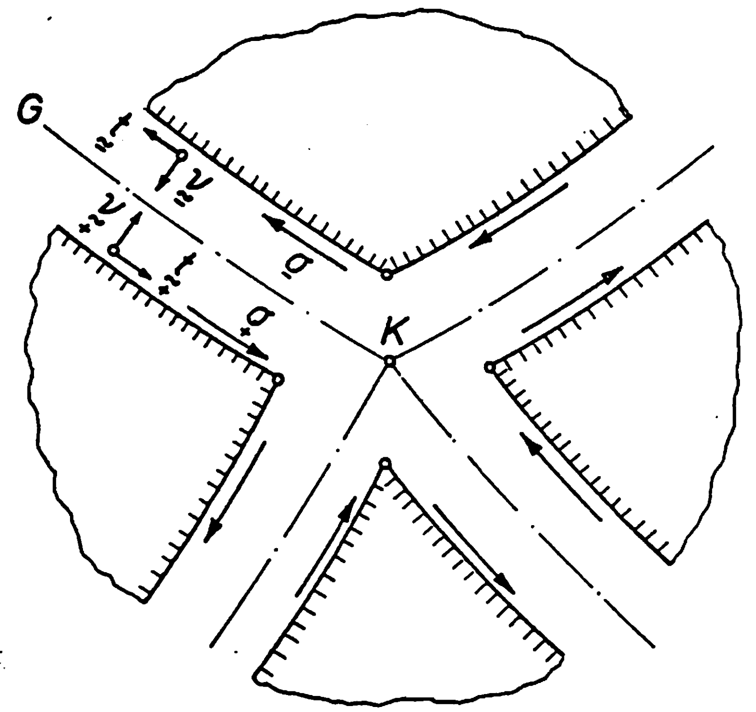

Fig. 2

Figure 2 shows a section of the plate devided into *finite elements*. The nodal point K is common point of all adjacent corner points. It also proves to be the point of intersection of all grid lines G. On two opposite sides + and -, the unit vectors of the triad $\underset{\sim}{t}$, $\underset{\sim}{\nu}$, $\underset{\sim}{n}$ obey the following relations:

$$\underset{+}{\underset{\sim}{t}} + \underset{-}{\underset{\sim}{t}} = (\underset{+}{t}^{\kappa} + \underset{-}{t}^{\kappa})\underset{\sim}{a}_{\kappa} = 0 \quad \Longrightarrow \quad \underset{+}{t}_{\kappa} + \underset{-}{t}_{\kappa} = 0 \qquad (3.1)$$

$$\underset{+}{\underset{\sim}{\nu}} + \underset{-}{\underset{\sim}{\nu}} = (\underset{+}{\nu}^{\kappa} + \underset{-}{\nu}^{\kappa})\underset{\sim}{a}_{\kappa} = 0 \quad \Longrightarrow \quad \underset{+}{\nu}_{\kappa} + \underset{-}{\nu}_{\kappa} = 0 \qquad (3.2)$$

$$\underset{+}{\underset{\sim}{n}} + \underset{-}{\underset{\sim}{n}} = 0 \qquad (3.3)$$

A surface tensor of the order $n (n \geq 0)$ may be given by $\underset{\sim}{T}$. Then the *discontinuity* or the *jump* of $\underset{\sim}{T}$ is represented in the basis of the positive boundary. Without loss of general validity, it suffices to show only one component of the complete jump.

$$\langle \underset{\sim}{T} \rangle = \cdots + \langle T_{t\nu\ldots t\ldots\nu} \rangle \underset{\underset{+}{\sim}}{t} \otimes \underset{\underset{+}{\sim}}{\nu} \otimes \ldots \otimes \underset{\underset{+}{\sim}}{t} \otimes \ldots \otimes \underset{\underset{+}{\sim}}{\nu} + \tag{3.4}$$

with

$$\langle T_{t\nu\ldots t\ldots\nu} \rangle = \left(\underset{+}{T}^{\alpha\beta\ldots\gamma\ldots\lambda} - \underset{-}{T}^{\alpha\beta\ldots\gamma\ldots\lambda} \right) \underset{+}{t}_{\alpha} \underset{+}{\nu}_{\beta} \ldots \underset{+}{t}_{\gamma} \ldots \underset{+}{\nu}_{\lambda} \tag{3.5}$$

The relations (3.1) and (3.2) transform the jump component of (3.5) into the final definition, which depends on the order n of the tensor.

$$\langle T_{t\nu\ldots t\ldots\nu} \rangle = \underset{+}{T}_{t\nu\ldots t\ldots\nu} \mp \underset{-}{T}_{t\nu\ldots t\ldots\nu} \quad \begin{cases} \text{„}-\text{"} : n \text{ even; } n = 0 \\ \text{„}+\text{"} : n \text{ odd} \end{cases} \tag{3.6}$$

where

$$\left. \begin{aligned} \underset{+}{T}_{t\nu\ldots t\ldots\nu} &= \underset{+}{T}^{\alpha\beta\ldots\gamma\ldots\lambda} \underset{+}{t}_{\alpha} \underset{+}{\nu}_{\beta} \ldots \underset{+}{t}_{\gamma} \ldots \underset{+}{\nu}_{\lambda} \\ \underset{-}{T}_{t\nu\ldots t\ldots\nu} &= \underset{-}{T}^{\alpha\beta\ldots\gamma\ldots\lambda} \underset{-}{t}_{\alpha} \underset{-}{\nu}_{\beta} \ldots \underset{-}{t}_{\gamma} \ldots \underset{-}{\nu}_{\lambda} \end{aligned} \right\} \tag{3.7}$$

Let $\underset{+}{T}$ and $\underset{+}{\psi}$ just as $\underset{-}{T}$ and $\underset{-}{\psi}$ be components, defined by (3.7), and let they belong to the tensors $\underset{\sim}{T}$ and $\underset{\sim}{\psi}$ of the order n and m respectively. The formula (3.6) specifies the corresponding discontinuities. Now, the sum $\underset{+}{T}\,\underset{+}{\psi} + \underset{-}{T}\,\underset{-}{\psi}$ shall be split up as follows:

$$\underset{+}{T}\,\underset{+}{\psi} + \underset{-}{T}\,\underset{-}{\psi} = \alpha (\underset{+}{T} + \underset{-}{T})(\underset{+}{\psi} + \underset{-}{\psi}) + (1-\alpha)(\underset{+}{T} - \underset{-}{T})(\underset{+}{\psi} - \underset{-}{\psi}) \tag{3.8}$$

The solution of this equation for the unknown factor α is

$$\alpha = \frac{1}{2} . \tag{3.9}$$

Furthermore, it can be concluded, that parenthetical expressions in (3.8), which belong to each other, are linearly independent. For instance

$$C_1 (\underset{+}{\psi} + \underset{-}{\psi}) + C_2 (\underset{+}{\psi} - \underset{-}{\psi}) = 0 \tag{3.10}$$

only holds for $c_1 = c_2 = 0$. On account of the independence of $\underset{+}{\psi}$ and $\underset{-}{\psi}$ this can be verified by the equation

$$(C_1 + C_2)\underset{+}{\psi} + (C_1 - C_2)\underset{-}{\psi} = 0 .$$

If line integrals have to be integrated by parts, then care must be taken separately of integrated terms at corners, where the curve is not smooth. This may happen at exterior or interior boundaries of the plate. An explanation for it will be given in chapter 4.

4. THE MODIFIED FUNCTIONAL OF HELLINGER-REISSNER

The foregoing expositions of chapter 2 have expressed, that the variational principle in displacements is related to the compatibility conditions and to the geometric boundary conditions, both as necessary restrictions. Since the geometric transitional conditions correspond to the geometric boundary conditions, those must be satisfied by geometric field quantities of the functional too.

Except for the constitutive equations, all side conditions are taken now into the functional by means of Lagrange mulitpliers. This shall be called *generalization* with regard to the compatibility and to geometric boundary conditions and *modification* with respect to geometric transitional conditions.

If one applies at the same time the transformation of Legendre to the functional, then the modified functional of Helliner-Reissner appears.

$$I_R = \left.\int_A [-W_C(N^{\alpha\beta},M^{\alpha\beta}) + M^{\alpha\beta}\chi_{\alpha\beta} + N^{\alpha\beta}\gamma_{\alpha\beta}]\, dA \right\} \text{LEGENDRE TRANSFORMATION}$$

$$\left.\begin{aligned}
&-\int_A p^* u_3 dA && -\int_{C_2} (P^*_{\nu\nu} u_\nu + P^*_{t\nu} u_t)\, ds \\
&-\int_{C_2} (P^*_{n\nu} u_3 + K^*_\nu \beta_\nu)\, ds && -[K^*_t u_3]_{C_2} \\
&-\int_\Gamma P^*_{n\nu} \tfrac{1}{2} (\underset{+}{u_3} + \underset{-}{u_3})\, d\sigma && -\int_\Gamma K^*_\nu \tfrac{1}{2} (\underset{+}{\beta_\nu} - \underset{-}{\beta_\nu})\, d\sigma
\end{aligned}\right\} \text{external load}$$

$$\left.\begin{aligned}
&-\int_A [\gamma_{\alpha\beta} - \tfrac{1}{2}(u_\alpha|_\beta + u_\beta|_\alpha + u_{3,\alpha} u_{3,\beta})]\, \underline{\hat{N}^{\alpha\beta}}\, dA \\
&-\int_A (\chi_{\alpha\beta} + u_3|_{\alpha\beta})\, \underline{\hat{M}^{\alpha\beta}}\, dA \\
&+\int_{C_1} (u^*_\nu - u_\nu)\, \underline{\hat{S}_\nu}\, ds \qquad +\int_{C_1} (u^*_t - u_t)\, \underline{\hat{S}_t}\, ds \\
&+\int_{C_1} (u^*_3 - u_3)\, \underline{\hat{V}}\, ds \qquad +\int_{C_1} (\beta^*_\nu - \beta_\nu)\, \underline{\hat{M}}\, ds \\
&+[(u^*_3 - u_3)\, \underline{\hat{T}_t}\,]_{C_1}
\end{aligned}\right\} \text{GENERALIZATION}$$

$$\left.\begin{aligned}
&-\int_\Gamma \langle u_\nu \rangle\, \underline{\hat{s}_\nu}\, d\sigma && -\int_\Gamma \langle u_t \rangle\, \underline{\hat{s}_t}\, d\sigma \\
&-\int_\Gamma \langle u_3 \rangle\, \underline{\hat{v}}\, d\sigma && \int_\Gamma \langle \beta_\nu \rangle\, \underline{\hat{m}}\, d\sigma \\
&-\Sigma \langle u_3 \rangle\, \underline{\hat{T}_G}
\end{aligned}\right\} \text{MODIFICATION} \qquad (4.1)$$

The total of interior boundaries of the plate is indicated by Γ. Parallel to the middle surface of the plate membrane forces do not exist. In vertical direction however the plate is stressed by the line load $P^*_{n\nu}$. Moreover, a bending moment K^*_ν is acting on Γ. Ohterwise, all geometric and static quantities are distinguished by an asterisk, if they are given from the beginning. All variables, which are supplied with the sign ^ have the meaning of Lagrange multipliers.

The multipiers $\hat{N}^{\alpha\beta}$ and $\hat{M}^{\alpha\beta}$ may be symmetrical as long as no contradiction arises.

The paranthetical expression on C_1 takes into account, that geometrical conditions at corner points are not fullfilled.

$$[(u_3^* - u_3)]_{C_1} = \sum_{i=1}^{N_1} [u_3^*(s_i) - u_3(s_i)]\, \hat{T}_1(s_i) \,\Big|_{C_1} \qquad (4.2)$$

The last sum of (3.1) eliminates all incompatibilities of the deflection u_3 on grid lines G at the nodes K (s. figure 3).

$$\sum \langle u_3 \rangle \hat{T}_G = \sum_K \sum_G \left[\underset{+}{u_3}(\underset{+}{\overset{}{\sigma}}{}^{K}_{G} - 0) - \underset{-}{u_3}(\underset{-}{\overset{}{\sigma}}{}^{K}_{G} + 0) \right] \hat{T}_G(\sigma^K_G) \qquad (4.3)$$

According to the stationary condition of the functional the first variation must vanish. Therefore, two expressions are integrated by parts after variation. This yields in view of all discontinuities by means of the integral theorems (C2.4) and (C2.7):

$$\int_A \hat{N}^{\alpha\beta} \tfrac{1}{2} \delta (u_\alpha|_\beta + u_\beta|_\alpha + u_{3,\alpha} u_{3,\beta})\, dA$$

$$= -\int_A \hat{N}^{\alpha\beta}|_\beta\, \delta u_\alpha\, dA \qquad + \oint_C (\hat{N}_{\nu\nu} \delta u_\nu + N_{t\nu} \delta u_t)\, ds$$

$$-\int_A \hat{N}^{\beta}|_\beta\, \delta u_3\, dA \qquad + \oint_C \hat{N}_\nu\, u_3\, ds$$

$$+\int_\Gamma (\underset{+}{\hat{N}_{\nu\nu}} \delta \underset{+}{u_\nu} + \underset{-}{N_{t\nu}} \delta \underset{-}{u_t})\, d\sigma \qquad + \int_\Gamma (\underset{-}{\hat{N}_{\nu\nu}} \delta \underset{-}{u_\nu} + \underset{-}{\hat{N}_{t\nu}} \delta \underset{-}{u_t})\, d\sigma$$

$$+\int_\Gamma \underset{+}{\hat{N}_\nu}\, \delta \underset{+}{u_3}\, d\sigma \qquad + \int_\Gamma \underset{-}{\hat{N}_\nu}\, \delta \underset{-}{u_3}\, d\sigma \qquad (4.4)$$

$$-\int_A \hat{M}^{\alpha\beta}\, \delta u_3|_{\alpha\beta}\, dA$$

$$= -\int_A \hat{M}^{\alpha\beta}|_{\alpha\beta}\, \delta u_3\, dA \qquad + \oint_C (\hat{M}_{\nu\nu}\,\delta\beta_\nu + \hat{P}_{m\nu}\,\delta u_3)\, ds$$

$$+\int_\Gamma (\underset{+}{\hat{M}}_{\nu\nu}\,\delta\underset{+}{\beta}_\nu + \underset{+}{\hat{P}}_{m\nu}\,\delta\underset{+}{u}_3)\, d\sigma \quad + \int_\Gamma (\underset{-}{\hat{M}}_{\nu\nu}\,\delta\underset{-}{\beta}_\nu + \underset{-}{\hat{P}}_{m\nu}\,\delta\underset{-}{u}_3)\, d\sigma$$

$$+\sum \underset{+}{\hat{M}}_{t\nu}\,\delta\underset{+}{u}_3 \quad - \sum \underset{-}{\hat{M}}_{t\nu}\,\delta u_3 \quad + \left[\hat{M}_{t\nu}\,\delta u_3\right]_C \tag{4.5}$$

The last two sums over all grid lines G include integrated terms. The minus sign of the second sum arises by the fact that on $\underset{+}{\sigma}$ a lower limit and on $\underset{-}{\sigma}$ an upper limit of integration exists at the node K (s. figure 2).

As abbreviations the formulae (2.16) and (2.17) were used. In accordance with definition (3.6) it is possible, to introduce the next transformations:

$$\underset{+}{N}_{\nu\nu}\,\delta\underset{+}{u}_\nu + \underset{-}{N}_{\nu\nu}\,\delta\underset{-}{u}_\nu$$

$$= \tfrac{1}{2}(\underset{+}{N}_{\nu\nu} + \underset{-}{N}_{\nu\nu})\,\delta\langle u_\nu\rangle + \langle N_{\nu\nu}\rangle\, \tfrac{1}{2}\,\delta(\underset{+}{u}_\nu - \underset{-}{u}_\nu) \tag{4.6}$$

$$\underset{+}{N}_{t\nu}\,\delta\underset{+}{u}_t + \underset{-}{N}_{t\nu}\,\delta\underset{-}{u}_t$$

$$= \tfrac{1}{2}(\underset{+}{N}_{t\nu} + \underset{-}{N}_{t\nu})\,\delta\langle u_t\rangle + \langle N_{t\nu}\rangle\, \tfrac{1}{2}\,\delta(\underset{+}{u}_t - \underset{-}{u}_t) \tag{4.7}$$

$$(\underset{+}{\hat{P}}_{m\nu} + \underset{+}{\hat{N}}_\nu)\,\delta\underset{+}{u}_3 + (\underset{-}{\hat{P}}_{m\nu} + \underset{-}{\hat{N}}_\nu)\,\delta\underset{-}{u}_3$$

$$= \tfrac{1}{2}\left[(\underset{+}{P}_{m\nu} + \underset{+}{N}_\nu) - (\underset{-}{P}_{m\nu} + \underset{-}{N}_\nu)\right]\delta\langle u_3\rangle + \langle P_{m\nu} + N_\nu\rangle\, \tfrac{1}{2}\,\delta(\underset{+}{u}_3 + \underset{-}{u}_3) \tag{4.8}$$

$$\underset{+}{M}_{\nu\nu}\,\delta\underset{+}{\beta}_\nu + \underset{-}{M}_{\nu\nu}\,\delta\underset{-}{\beta}_\nu$$

$$= \tfrac{1}{2}(\underset{+}{M}_{\nu\nu} + \underset{-}{M}_{\nu\nu})\,\delta\langle \beta_\nu\rangle + \langle M_{\nu\nu}\rangle\, \tfrac{1}{2}\,\delta(\underset{+}{\beta}_\nu - \underset{-}{\beta}_\nu) \tag{4.9}$$

For the integrated terms one receives the transformation

$$\underset{+}{M}_{t\nu}\,\delta\underset{+}{u}_3 - \underset{-}{M}_{t\nu}\,\delta\underset{-}{u}_3$$

$$= \tfrac{1}{2}(\underset{+}{M}_{t\nu} + \underset{-}{M}_{t\nu})\,\delta\langle u_3\rangle + \langle M_{t\nu}\rangle\, \tfrac{1}{2}\,\delta(\underset{+}{u}_3 + \underset{-}{u}_3)\,. \tag{4.10}$$

If all expressions of (4.4) to (4.10) are introduced into (4.1), the first variation of the functional attains its final form. The linear independence of all quantities subject to variation is assured by the considerations of (3.10) made for equation (3.8). This leads to the *Euler-Lagrange-equations* and to all *boundary* and *interelement boundary conditions*. Particularly, the physical meaning of all *Lagrange multipliers* is specified.

$$\delta I_R = \left.-\int_A \left(\frac{\partial W_C}{\partial N^{\alpha\beta}} - \gamma_{\alpha\beta}\right)\delta N^{\alpha\beta} dA - \int_A \left(\frac{\partial W_C}{\partial M^{\alpha\beta}} - \varkappa_{\alpha\beta}\right)\delta M^{\alpha\beta} dA \quad\right\} \text{constit. equations}$$

$$\left.\begin{aligned}
&-\int_A \left(\gamma_{\alpha\beta} - \tfrac{1}{2}\left(u_\alpha|_\beta + u_\beta|_\alpha + u_{3,\alpha}u_{3,\beta}\right)\right)\delta \hat{N}^{\alpha\beta} dA \\
&-\int_A \left(\varkappa_{\alpha\beta} + u_3|_{\alpha\beta}\right)\delta \hat{M}^{\alpha\beta} dA
\end{aligned}\right\} \text{strain-displ.-relations}$$

$$\left.-\int_A \hat{N}^{\alpha\beta}|_\beta\, \delta u_\alpha dA - \int_A \left(\hat{M}^{\alpha\beta}|_{\alpha\beta} + \hat{N}^\beta|_\beta - p^*\right)\delta u_3 dA \quad\right\} \text{EQUILIBRIUM}$$

$$\left.\begin{aligned}
&-\int_{C_2} \left(P^*_{\nu\nu} - \hat{N}_{\nu\nu}\right)\delta u_\nu ds &&-\int_{C_2} \left(P^*_{t\nu} - \hat{N}_{t\nu}\right)\delta u_t ds \\
&-\int_{C_2} \left(P^*_{n\nu} - (\hat{P}_{n\nu} + \hat{N}_\nu)\right)\delta u_3 ds && \\
&-\int_{C_2} \left(K^*_\nu - \hat{M}_{\nu\nu}\right)\delta\beta_\nu ds &&- \left[\left(K^*_t - \hat{M}_{t\nu}\right)\delta u_3\right]_{C_2}
\end{aligned}\right\} \text{statical boundary conditions}$$

$$\left.\begin{aligned}
&+\int_{C_1} \left(u^*_t - u_t\right)\delta \hat{S}_t ds &&+\int_{C_1} \left(u^*_\nu - u_\nu\right)\delta \hat{S}_\nu ds \\
&+\int_{C_1} \left(u^*_3 - u_3\right)\delta \hat{V} ds &&+\left[\left(u^*_3 - u_3\right)\delta \hat{T}_t\right]_{C_1} \\
&+\int_{C_1} \left(\beta^*_\nu - \beta_\nu\right)\delta \hat{M}_{\nu\nu} ds &&
\end{aligned}\right\} \text{geom. boundary conditions}$$

$$\left.\begin{aligned}
&+\int_\Gamma \langle \hat{N}_{\nu\nu}\rangle \tfrac{1}{2}\delta\left(\underset{+}{u_\nu} - \underset{-}{u_\nu}\right) d\sigma \qquad +\int_\Gamma \langle \hat{N}_{t\nu}\rangle \tfrac{1}{2}\delta\left(\underset{+}{u_t} - \underset{-}{u_t}\right) d\sigma \\
&-\int_\Gamma \left(P^*_{n\nu} - \langle \hat{P}_{n\nu} + \hat{N}_\nu\rangle\right) \tfrac{1}{2}\delta\left(\underset{+}{u_3} + \underset{-}{u_3}\right) d\sigma \\
&-\int_\Gamma \left(K^*_\nu - \langle \hat{M}_{\nu\nu}\rangle\right) \tfrac{1}{2}\delta\left(\underset{+}{\beta_\nu} - \underset{-}{\beta_\nu}\right) d\sigma \\
&-\sum \langle \hat{M}_{t\nu}\rangle \tfrac{1}{2}\delta\left(\underset{+}{u_3} - \underset{-}{u_3}\right)
\end{aligned}\right\} \text{statical interelement bound. cond.}$$

$$\left.\begin{array}{ll}
-\int_{\Gamma}\langle u_\nu\rangle\,\delta\hat{s}_\nu\,d\sigma & -\int_{\Gamma}\langle u_t\rangle\,\delta\hat{s}_t\,d\sigma \\
-\int_{\Gamma}\langle u_3\rangle\,\delta\hat{v}\,d\sigma & -\int_{\Gamma}\langle\beta_\nu\rangle\,\delta\hat{m}\,d\sigma \\
-\sum\langle u_3\rangle\,\delta\hat{T}_G &
\end{array}\right\}\begin{array}{l}\textit{geometrical}\\ \textit{interelement}\\ \textit{bound. cond.}\end{array}$$

$$\left.\begin{array}{l}
-\int_A(\hat{N}^{\alpha\beta}-N^{\alpha\beta})\,\delta\gamma_{\alpha\beta}\,dA \\
-\int_A(\hat{M}^{\alpha\beta}-M^{\alpha\beta})\,\delta\varkappa_{\alpha\beta}\,dA \\
-\int_{C_1}(\hat{S}_\nu-\hat{N}_{\nu\nu})\,\delta u_\nu\,ds \\
-\int_{C_1}(\hat{S}_t-\hat{N}_{t\nu})\,\delta u_t\,ds \\
-\int_{C_1}[\hat{V}-(\hat{P}_{n\nu}+\hat{N}_\nu)]\,\delta u_3\,ds \\
-\int_{C_1}(\hat{M}-\hat{M}_{\nu\nu})\,\delta\beta_\nu\,ds \\
-[(\hat{T}_t-\hat{M}_{t\nu})\,\delta u_3]_{C_1} \\
-\int_{\Gamma}[\hat{s}_\nu-\tfrac{1}{2}(\underset{+}{\hat{N}_{\nu\nu}}+\underset{-}{\hat{N}_{\nu\nu}})]\,\delta\langle u_\nu\rangle\,d\sigma \\
-\int_{\Gamma}[\hat{s}_t-\tfrac{1}{2}(\underset{+}{\hat{N}_{t\nu}}+\underset{-}{\hat{N}_{t\nu}})]\,\delta\langle u_t\rangle\,d\sigma \\
-\int_{\Gamma}\big(\hat{v}-\tfrac{1}{2}[(\hat{P}_{n\nu}+\hat{N}_\nu)_+-(\hat{P}_{n\nu}+\hat{N}_\nu)_-]\big)\,\delta\langle u_3\rangle\,d\sigma \\
-\int_{\Gamma}[\hat{m}-\tfrac{1}{2}(\underset{+}{\hat{M}_{\nu\nu}}+\underset{-}{\hat{M}_{\nu\nu}})]\,\delta\langle\beta_\nu\rangle\,d\sigma \\
-\sum[\hat{T}_G-\tfrac{1}{2}(\underset{+}{\hat{M}_{t\nu}}-\underset{-}{\hat{M}_{t\nu}})]\,\delta\langle u_3\rangle
\end{array}\right\}\begin{array}{l}\textit{Lagrange -}\\ \textit{Factors}\end{array} \qquad (4.11)$$

5. A MIXED HU-WASHIZU FUNCTIONAL

5.1. The derivation of the Hu-Washizu functional

It is possible, to derive the Hu-Washizu functional from the principle of the stationary value of potential energy (2.7) with subsidiary conditions (2.9) and (2.10). With regard to all terms of (2.7), which are influenced by stretching of the plate middle surface, the Hu-Washizu functional is developed, if the first side condition of (2.9) and the geometric boundary conditions of membrane displacements in (2.10) are removed. The corresponding Lagrange factors are defined already by consideration of the modified Reissner functional according to (4.11). Applying also Legendre's transforamtion (2.14) with respect to the bending part, it follows

$$
\begin{aligned}
I_H = & \int_A W_M(\gamma_{\alpha\beta})\,dA - \int_A W_{CB}(M^{\alpha\beta})\,dA - \int_A p^* u_3\,dA \\
& + \int_A M^{\alpha\beta}\varkappa_{\alpha\beta}\,dA \\
& - \int_A N^{\alpha\beta}\left[\gamma_{\alpha\beta} - \tfrac{1}{2}\left(u_\alpha|_\beta + u_\beta|_\alpha + u_{3,\alpha}u_{3,\beta}\right)\right] dA \\
& - \int_{C_2} \left(P^*_{\nu\nu} u_\nu + P^*_{t\nu} u_t\right) ds - \int_{C_2} P^*_{n\nu} u_3\,ds - \int_{C_2} K^*_\nu \beta_\nu\,ds \\
& + \left[K^*_t u_3\right]_{C_2} \\
& + \int_{C_1} (u^*_t - u_t) N_{t\nu}\,ds + \int_{C_1} (u^*_\nu - u_\nu) N_{\nu\nu}\,ds
\end{aligned}
\qquad (5.1.1)
$$

The preceding operations define the functional (5.1.1) in the sense of Hu-Washizu only with respect to the membrane part of the von Kármán-plate. Now, after integration by parts, applied to membrane displacements in the fifth integral, one receives another form of the functional.

$$
\begin{aligned}
I_H = & \int_A W_M(\gamma_{\alpha\beta})\,dA - \int_A W_{CB}(M^{\alpha\beta})\,dA - \int_A p^* u_3\,dA \\
& + \int_A M^{\alpha\beta}\varkappa_{\alpha\beta}\,dA - \int_A N^{\alpha\beta}\left(\gamma_{\alpha\beta} - \tfrac{1}{2}u_{3,\alpha}u_{3,\beta}\right) dA \\
& - \int_A N^{\alpha\beta}|_\beta\, u_\alpha\,dA
\end{aligned}
$$

$$-\int_{C_2} (P_{n\nu}^* u_3 + K_\nu^* \beta_\nu)\, ds - [\, K_t^* u_3 \,]_{C_2}$$

$$-\int_{C_2} (P_{\nu\nu}^* - N_{\nu\nu})\, u_\nu \, ds \quad -\int_{C_2} (P_{t\nu}^* - N_{t\nu})\, u_t \, ds$$

$$+\int_{C_1} (N_{t\nu}\, u_t^* + N_{\nu\nu}\, u_\nu^*)\, ds \tag{5.1.2}$$

In this functional three side conditions of the state of membrane stresses are forthcoming:

Equilibrium equations in A and statical boundary conditions on C_2:

In A:
$$N^{\alpha\beta}|_\beta = 0 \tag{5.1.3}$$

On C_2:
$$\left.\begin{aligned} P_{\nu\nu}^* - N_{\nu\nu} &= 0 \\ P_{t\nu}^* - N_{t\nu} &= 0 \end{aligned}\right\} \tag{5.1.4}$$

Introducing Airy's stress function F, these conditions can be satisfied. For later convenience it is suitable to use a representation according to [44], and that is

$$N^{\alpha\beta} = \epsilon^{\beta\lambda} \chi^\alpha|_\lambda \, . \tag{5.1.5}$$

$$\chi^\alpha = \epsilon^{\alpha\rho} F_{,\rho} \, . \tag{5.1.6}$$

5.2. The transformation of the membrane boundary integral

The difficulty, which arises by introducing Airy's stress function is now to transform the last integral of (5.1.2) in a proper way. The prescribed boundary displacements u_α^* shall be differentiable as far as needed. With (5.1.5) it follows at first

$$\int_{C_1} (N_{\nu\nu}\, u_\nu^* + N_{t\nu}\, u_t^*)\, ds = \int_{C_1} N^{\alpha\beta} \nu_\beta\, u_\alpha^* \, ds$$

$$= \int_{C_1} \chi^\alpha|_\lambda\, t^\lambda u_\alpha^* \, ds \, . \tag{5.2.1}$$

Since covariant and partial derivatives of the scalar $\chi^\alpha u_\alpha^*$ are equal to each other, the next relation holds.

$$\chi^{\alpha}/_{\lambda}\, t^{\lambda} u_{\alpha}^{*} = (\chi^{\alpha} u_{\alpha}^{*})_{,s} - \chi^{\alpha} u_{\alpha}^{*}/_{\lambda}\, t^{\lambda} \tag{5.2.2}$$

This relation is used for partial integration of (5.2.1).

$$\int_{C_1} \chi^{\alpha}/_{\lambda}\, t^{\lambda} u_{\alpha}^{*}\, ds = -\int_{C_1} \chi^{\alpha} u_{\alpha}^{*}/_{\lambda}\, t^{\lambda}\, ds + (\chi^{\alpha} u_{\alpha}^{*})_{C_1} \tag{5.2.3}$$

The integrated term states the difference of values on both limits of the intervall C_1, over which was integrated from s_o to s_1.

$$(\chi^{\alpha} u_{\alpha}^{*})_{C_1} = \chi^{\alpha}(s_1)\, u_{\alpha}^{*}(s_1) - \chi^{\alpha}(s_o)\, u_{\alpha}^{*}(s_o) \tag{5.2.4}$$

Instead of covariant derivatives of the components of boundary displacements a well-known decomposition is used.

$$\begin{aligned} u_{\alpha}^{*}/_{\lambda} &= \tfrac{1}{2}(u_{\alpha}^{*}/_{\lambda} + u_{\lambda}^{*}/_{\alpha}) + \tfrac{1}{2}(u_{\alpha}^{*}/_{\lambda} - u_{\lambda}^{*}/_{\alpha}) \\ &= \Theta_{\alpha\lambda}^{*} + \omega_{\alpha\lambda}^{*} \end{aligned} \tag{5.2.5}$$

Herein, $\omega_{\alpha\lambda}$ are components of the tensor of *infinitesimal rotation* about the normal $\underset{\sim}{n}$.

$$\omega_{\alpha\lambda} = \tfrac{1}{2}(u_{\alpha}/_{\lambda} - u_{\lambda}/_{\alpha}) \tag{5.2.6}$$

Formula (B1.10) transforms the tensor (5.2.5) subsequently to the coordinate system of the boundary.

$$\begin{aligned} u_{\alpha}^{*}/_{\lambda} = {} & \Theta_{tt}^{*}\, t_{\alpha} t_{\lambda} + (\Theta_{t\nu}^{*} - \omega_{t\nu}^{*})\, t_{\alpha} \nu_{\lambda} \\ & + (\Theta_{\nu t}^{*} - \omega_{\nu t}^{*})\, \nu_{\alpha} t_{\lambda} + \Theta_{\nu\nu}^{*}\, \nu_{\alpha} \nu_{\lambda} \end{aligned} \tag{5.2.7}$$

Considering the orthogonality relations of (A1.11) a scalar multiplication of (5.2.7) by t^{λ} leads to

$$u_{\alpha}^{*}/_{\lambda}\, t^{\lambda} = \Theta_{tt}^{*}\, t_{\alpha} + (\Theta_{\nu t}^{*} - \omega_{\nu t}^{*})\, \nu_{\alpha} . \tag{5.2.8}$$

Consequently, after further multiplication by χ^{α} the integrand of the right hand side of (5.2.3) is represented in the vector base of the boundary system.

$$-\chi^{\alpha} u_{\alpha}^{*}/_{\lambda}\, t^{\lambda} = -\chi_t\, \Theta_{tt}^{*} - \chi_{\nu}\, (\Theta_{\nu t}^{*} - \omega_{\nu t}^{*}) \tag{5.2.9}$$

Following definition (5.1.6) the physical components in the boundary system are

$$\left.\begin{aligned} \chi_t &= \chi^\alpha t_\alpha = -F_{,\nu} \\ \chi_\nu &= \chi^\alpha \nu_\alpha = F_{,t} \end{aligned}\right\} \tag{5.2.10}$$

such that (5.2.9) changes to

$$-\chi^\alpha u_\alpha^*/_\lambda t^\lambda = F_{,\nu}\, \Theta_{tt}^* - F_{,t}\,(\Theta_{\nu t}^* - \omega_{\nu t}^*)\,. \tag{5.2.11}$$

Because of $F_{,t} = F_{,s}$ in view of (B2.5), the second term of the right hand side of (5.2.11) can be integrated once more. Hence it follows

$$-\int_{C_1} \chi^\alpha u_\alpha^*/_\lambda t^\lambda ds = \int_{C_1} [F(D_\nu^* - \varphi^*)_{,s} + F_{,\nu} D_t^*]ds + [D_\nu^* F]_{C_1} - (\varphi^* F)_{,s}\,. \tag{5.2.12}$$

At all corners of the boundary C_1 the whole lot of discontinuities

$$[D_\nu^* F]_{C_1} = \sum_1^N [\Theta_{\nu t}^*(s_i+0) - \Theta_{\nu t}^*(s_i-0)]\, F(s_i)$$

has to be taken into consideration.
With $\omega_{\nu t}^*$ as infinitesimal rotation on C_1 the abbreviations, used in (5.2.12), stand for

$$\left.\begin{aligned} D_t^* &= \Theta_{tt}^* \\ D_\nu^* &= \Theta_{\nu t}^* \end{aligned}\right\} \tag{5.2.13}$$

$$\begin{aligned} \omega_{\nu t}^* &= \omega_{\alpha\beta}^* \nu^\alpha t^\beta = \tfrac{1}{2}(\omega_{\alpha\beta}^* \nu^\alpha t^\beta + \omega_{\beta\alpha}^* \nu^\beta t^\alpha) \\ &= \tfrac{1}{2}\omega_{\alpha\beta}^* \delta_{\varrho\lambda}^{\alpha\beta} = \tfrac{1}{2}\epsilon^{\alpha\beta}\omega_{\alpha\beta}^* = \varphi^*\,. \end{aligned} \tag{5.2.14}$$

If $\omega_{\alpha\beta}^*$ is substituted by (5.2.5), there is another relation for φ^*.

$$\varphi^* = \tfrac{1}{2}\epsilon^{\alpha\beta}(\Theta_{\alpha\beta}^* - u_\alpha^*/_\beta) = \tfrac{1}{2}\epsilon^{\alpha\beta} u_\beta/_\alpha \tag{5.2.15}$$

Since φ^* is continuous at corner points, only values on the limits of the intervall are retained.

$$(\varphi^* F)_{C_1} = \varphi^*(s_1)F(s_1) - \varphi^*(s_0)F(s_0) \tag{5.2.16}$$

Finally the complete transformation of the boundary integral (5.2.1) is finished together with (5.2.3) and (5.2.12).

$$\int_{C_1} N^{*\alpha\beta}\nu_\beta u_\alpha^* \, ds = -\int_{C_1} \left[F(\varphi^* - D_\nu^*)_{,s} + \chi_t D_t^* \right] ds$$

$$+ \left[D_\nu^* F \right]_{C_1} + (\chi^* u_\alpha^*)_{C_1} - (\varphi^* F)_{C_1} \qquad (5.2.17)$$

5.3. About geometrical interpretation of the transformed boundary integral

To the result of (5.2.17) a physical interpretation is possible, if length and direction of $\mathbf{t}$ and $\boldsymbol{\nu}$ are examined after deformation. In the deformed configuration they may be transposed to the vectors $\bar{\mathbf{t}}$ and $\bar{\boldsymbol{\nu}}$. The vector of position $\bar{\mathbf{r}}$ in the state of deformation is related to the undeformed configuration by the displacement vector $\mathbf{u}$.

$$\bar{\mathbf{r}} = \mathbf{r} + \mathbf{u} \qquad (5.3.1)$$

Differentiating along parameters s and ν using (2.6), (5.2.6) and (A3.4), this yields with $b_{\alpha\beta} = 0$

$$\begin{aligned} \bar{\mathbf{t}} &= \mathbf{t} + \mathbf{u}_{,s} \\ &= \mathbf{t} + (u_\alpha|_\beta \, \mathbf{a}^\alpha + u_{3,\beta}\, \mathbf{n})\, t^\beta \\ &= (1 + \theta_{tt})\, \mathbf{t} + (\theta_{\nu t} - \omega_{\nu t})\, \boldsymbol{\nu} + \varphi_t \, \mathbf{n} \end{aligned} \qquad (5.3.2)$$

$$\begin{aligned} \bar{\boldsymbol{\nu}} &= \boldsymbol{\nu} + \mathbf{u}_{,\nu} \\ &= \boldsymbol{\nu} + (u_\alpha|_\beta \, \mathbf{a}^\alpha + u_{3,\beta}\, \mathbf{n})\, \nu^\beta \\ &= (1 + \theta_{\nu\nu})\, \boldsymbol{\nu} + (\theta_{\nu t} + \omega_{\nu t})\, \mathbf{t} + \varphi_\nu \, \mathbf{n} \, . \end{aligned} \qquad (5.3.3)$$

In addition, the transformation rule (B1.2) has come into account.
It can be observed by (5.3.2) together with (5.3.3) that the unit vectors $\mathbf{t}$ and $\boldsymbol{\nu}$ perform a *rigid body rotation* by $\omega_{\nu t}$ and then rotate opposite to each other by $\theta_{\nu t}$. The right angle between $\mathbf{t}$ and $\boldsymbol{\nu}$ becomes smaller by the qunatitiy $2\theta_{\nu t}$.
Consequently, a boundary element ds suffers an *effective rotation* of

$$\omega = \varphi - \theta_{\nu t} \, . \qquad (5.3.4)$$

Moving into direction of s, the change of ω admits a clear interpretation too. First, the *gradient of rotation* is by definition

$$\varphi|_\mu = \varkappa_{\mu 3} = \tfrac{1}{2}\,\epsilon^{\alpha\beta}\,u_\beta|_{\alpha\mu} . \tag{5.3.5}$$

In Euclidean spaces the sequence of covariant derivatives is interchangeable. Using this fact and regarding also the properties of the $\underset{\sim}{\epsilon}$-tensor, it follows from (5.3.5)

$$\varkappa_{\mu 3} = \tfrac{1}{2}\,\epsilon^{\alpha\beta}\left(u_\beta|_{\mu\alpha} + u_\mu|_{\beta\alpha}\right) = \epsilon^{\alpha\beta}\,\Theta_{\beta\mu}|_\alpha . \tag{5.3.6}$$

As abbreviation the tensor $T^{\alpha}_{\cdot\mu}$ is introduced for the present.

$$T^{\alpha}_{\cdot\mu} = \epsilon^{\alpha\beta}\,\Theta_{\beta\mu} \tag{5.3.7}$$

For the transformation of its covariant derivative formula (B2.14) is available.

$$T^{\alpha}_{\cdot\mu}|_\alpha = \left(T_{tt,t} + T_{\nu t,\nu}\right) t_\mu + \left(T_{t\nu,t} + T_{\nu\nu,\nu}\right)\nu_\mu$$

Multiplication by t^μ yields according to (5.3.5) with (5.3.7)

$$\varphi_{,s} = T_{tt,t} + T_{\nu t,\nu} = -\Theta_{\nu t,t} + \Theta_{tt,\nu} . \tag{5.3.8}$$

This expression is part of the *change of inplane curvature* [35] which is denoted by κ_{s3e}. Here, referring to [37],(6.5.9)] it is signified by k_{nt} and specified by the name *effective curvature* (*Ersatzkrümmung* [45]).

$$\begin{aligned} k_{nt} &= \varphi_{,s} - \Theta_{\nu t,s} \\ &= -\Theta_{\nu t,t} + \Theta_{tt,\nu} - \Theta_{\nu t,s} \\ &= -2\Theta_{\nu t,s} + \Theta_{tt,\nu} - \varkappa_t\left(\Theta_{\nu\nu} - \Theta_{tt}\right) \end{aligned} \tag{5.3.9}$$

Formula (B2.8) has been used.

In relation (6.5.9) of [37], there must be set $\kappa_\nu = 0$ and instead of the nonlinear strain tensor the linear one has to be introduced, since all membrane boundary conditions are geometrically linear. By the way, it may be allowed to keep the present definition of mathematical sign for k_{nt} in contrast to that from [37].

5.4. The final form of the Hu-Washizu functional

On account of the transformed boundary integral (5.2.17) the definition of the functional (5.1.2) can be delivered from membrane displacements. The last two integrated terms of (5.2.17) are not needed. Now, the functional (5.1.2) appears in the following form:

$$
\begin{aligned}
I_H = & \int_A W_M\left(\Theta_{\alpha\beta} + \tfrac{1}{2} u_{3,\alpha} u_{3,\beta}\right) dA \\
& + \int_A M^{\alpha\beta} \kappa_{\alpha\beta}\, dA - \int_A N^{\alpha\beta} \Theta_{\alpha\beta}\, dA - \int_A p^* u_3\, dA \\
& - \int_{C_1} P^*_{n\nu} u_3\, ds - \int_{C_2} K^*_\nu \beta_\nu\, ds - \left[K^*_t u_3 \right]_{C_2} \\
& - \int_{C_1} k^*_{nt} F\, ds - \int_{C_1} D^*_t \chi_t\, ds + \left[D^*_\nu F \right]_{C_1}
\end{aligned}
\qquad (5.4.1)
$$

The subsidiary conditions are defined as follows:

In A:
$$
\left.
\begin{aligned}
\kappa_{\alpha\beta} + u_3|_{\alpha\beta} &= 0 \\
N^{\alpha\beta} - \epsilon^{\alpha\rho}\epsilon^{\beta\lambda} F|_{\rho\lambda} &= 0
\end{aligned}
\right\}
\qquad (5.4.2)
$$

On C_1:
$$
\left.
\begin{aligned}
u_3^* - u_3 &= 0 \\
\beta_\nu^* - \beta_\nu &= 0
\end{aligned}
\right\}
\qquad (5.4.3)
$$

On C_2:
$$
\left.
\begin{aligned}
F^* - F &= 0 \\
\chi_t^* - \chi_t &= 0
\end{aligned}
\right\}
\qquad (5.4.4)
$$

It is remarked that in functional (5.4.1) the linear strain tensor $\Theta_{\alpha\beta}$ is to treat as independent field variable. Another possibility arises, if one chooses the nonlinear strain tensor $\gamma_{\alpha\beta}$ for free variation.

$$
\begin{aligned}
I_H = & \int_A W_M(\gamma_{\alpha\beta})\, dA - \int_A W_{CB}(M^{\alpha\beta})\, dA - \int_A p^* u_3\, dA \\
& + \int_A M^{\alpha\beta} \kappa_{\alpha\beta}\, dA - \int_A N^{\alpha\beta}\left(\gamma_{\alpha\beta} - \tfrac{1}{2} u_{3,\alpha} u_{3,\beta}\right) dA
\end{aligned}
$$

$$-\int_{C_2} (P_{n\nu}^{*} u_3 + K_{\nu}^{*} \beta_\nu) ds \quad - [K_t^{*} u_3]_{C_2}$$

$$-\int_{C_1} (k_{nt}^{*} F + D_t^{*} \chi_t) ds \quad + [D_\nu^{*} F]_{C_1} \qquad (5.4.5)$$

All side conditions of (5.4.2) to (5.4.4) are not changed.

6. THE GENERALIZATION AND MODIFICATION OF THE HU-WASHIZU FUNCTIONAL

In order to generalize the Hu-Washizu functional, all side conditions are added to the functional by means of Lagrange multipliers. The modification takes place in the same way as it was done in chapter 4 for the Reissner functional by removing all geometric and static interelement boundary conditions. Here, those quantities distinguished by a hat $\hat{}$ are Lagrange factors again. Also, the Lagrange factor $\hat{\theta}_{\alpha\beta}$ shall be symmetrical as long as no contradiction turns up.

$$
\begin{aligned}
I_H = & \left.\begin{aligned}
& \int_A W_M(\gamma_{\alpha\beta})\,dA - \int_A W_{CB}(M^{\alpha\beta})\,dA - \int_A p^* u_3\,dA \\
& + \int_A M^{\alpha\beta}\chi_{\alpha\beta}\,dA - \int_A N^{\alpha\beta}(\gamma_{\alpha\beta} - \tfrac{1}{2}u_{3,\alpha}u_{3,\beta})\,dA \\
& - \int_{C_2} P^*_{n\nu}u_3\,ds - \int_{C_2} K^*_\nu \beta_\nu\,ds - [K^*_t u_3]_{C_2} \\
& - \int_C k^*_{nt}F\,ds - \int_C D^*_t X_t\,ds + [D^*_\nu F]_{C_1}
\end{aligned}\right\}\ \text{original functional} \\[2ex]
& \left.\begin{aligned}
& - \int_\Gamma P^*_{n\nu}\tfrac{1}{2}(\underset{+}{u_3} + \underset{-}{u_3})\,d\sigma - \int_\Gamma K^*_\nu \tfrac{1}{2}(\underset{+}{\beta_\nu} - \underset{-}{\beta_\nu})\,d\sigma \\
& - \int_\Gamma k^*_{nt}\tfrac{1}{2}(\underset{+}{F} + \underset{-}{F})\,d\sigma - \int_\Gamma D^*_t \tfrac{1}{2}(\underset{+}{X_t} - \underset{-}{X_t})\,d\sigma
\end{aligned}\right\}\ \text{line loads} \\[2ex]
& \left.\begin{aligned}
& - \int_A (\chi_{\alpha\beta} + u_3|_{\alpha\beta})\underline{\hat{M}^{\alpha\beta}}\,dA - \int_A (N^{\alpha\beta} - \varepsilon^{\alpha\varrho}\varepsilon^{\beta\lambda}F|_{\varrho\lambda})\,\underline{\hat{\theta}_{\alpha\beta}}\,dA \\
& + \int_{C_1} (u_3^* - u_3)\underline{\hat{V}}\,ds + \int_{C_1} (\beta_\nu^* - \beta_\nu)\underline{\hat{M}}\,ds \\
& + \int_{C_2} (F^* - F)\underline{\hat{R}}\,ds + \int_{C_2} (X_t^* - X_t)\underline{\hat{D}}\,ds \\
& + [(u_3^* - u_3)\underline{\hat{T}_t}]_{C_1} - [(F^* - F)\underline{\hat{R}_t}]_{C_2}
\end{aligned}\right\}\ \text{GENERALIZATION} \\[2ex]
& \left.\begin{aligned}
& - \int_\Gamma \langle u_3\rangle \underline{\hat{v}}\,d\sigma - \int_\Gamma \langle \beta_\nu\rangle \underline{\hat{m}}\,d\sigma \\
& - \int_\Gamma \langle F\rangle \underline{\hat{f}}\,d\sigma - \int_\Gamma \langle X_t\rangle \underline{\hat{s}}\,d\sigma \\
& - \sum \langle u_3\rangle \underline{\hat{T}_G} - \sum \langle F\rangle \underline{\hat{S}_G}
\end{aligned}\right\}\ \text{MODIFICATION}
\end{aligned}
\qquad (6.1)
$$

Singular conditions at corner points on C_2 exist for the Airy stress function completely in analogy to those of the plate deflection u_3.

$$[(F^* - F)\hat{R}_t]_{C_2} = \sum_{i=1}^{N_2} [F^*(s_i) - F(s_i)]\hat{R}_t(s_i) /_{C_2} \qquad (6.2.)$$

Reference was taken already to the corresponding expression for u_3 in chapter 4.

The line loads $P^*_{n\nu}$ and K^*_ν of the plate discover their analogy by a distributed change of in-plane curvature k^*_{nt} and by a distributed stretching quantity D^*_t.

By the sum

$$\sum \langle F \rangle \hat{S}_G = \sum_K \sum_G [\underset{+}{F}(\underset{K}{\sigma_G} - 0) - \underset{-}{F}(\underset{K}{\sigma_G} + 0)]\hat{S}_G(\underset{K}{\sigma_G}) \qquad (6.3)$$

all incompatibilities of the stress function on grid lines G at the nodes K are included.

The Hu-Washizu functional attains a stationary value, if the first variation vanishes. Before describing the final form of the variation, some more integral transformations are needed. Two of them are prepared already by the formulae (4.4) to (4.10) and they may be comprised once more for the sake of completeness.

$$
\begin{aligned}
&-\int_A \hat{M}^{\alpha\beta} \delta u_3|_{\alpha\beta}\, dA \, . \\
&= -\int_A \hat{M}^{\alpha\beta}|_{\alpha\beta}\, \delta u_3\, dA + \oint_C (\hat{M}_{\nu\nu}\, \delta\beta_\nu + \hat{P}_{n\nu}\, \delta u_3)\, ds \\
&+ \int_\Gamma \mp\tfrac{1}{2}(\underset{+}{\hat{M}_{\nu\nu}} + \underset{-}{\hat{M}_{\nu\nu}})\,\delta\langle\beta_\nu\rangle\, d\sigma + \int_\Gamma \langle\hat{M}_{\nu\nu}\rangle \mp\tfrac{1}{2}\,\delta(\underset{+}{\beta_\nu} - \underset{-}{\beta_\nu})\, d\sigma \\
&+ \int_\Gamma \mp\tfrac{1}{2}(\underset{+}{\hat{P}_{n\nu}} - \underset{-}{\hat{P}_{n\nu}})\,\delta\langle u_3\rangle\, d\sigma + \int \langle P_{n\nu}\rangle \mp\tfrac{1}{2}\,\delta(\underset{+}{u_3} + \underset{-}{u_3})\, d\sigma \\
&+ \sum \mp\tfrac{1}{2}(\underset{+}{\hat{M}_{t\nu}} + \underset{-}{\hat{M}_{t\nu}})\,\delta\langle u_3\rangle + \sum \langle M_{t\nu}\rangle \mp\tfrac{1}{2}\,\delta(\underset{+}{u_3} + \underset{-}{u_3}) \\
&+ [\hat{M}_{t\nu}\, \delta u_3]_C
\end{aligned}
\qquad (6.4)
$$

$$
\begin{aligned}
\int_A \hat{N}^{\alpha\beta} u_{3,\beta}\, \delta u_{3,\alpha}\, dA &= \int_A \hat{N}^{\alpha} \delta u_{3,\alpha}\, dA \\
-\int_A \hat{N}^{\alpha}|_{\alpha}\, \delta u_3\, dA \qquad & + \int_\Gamma \hat{N}_\nu\, \delta u_3\, d\sigma \\
+\int_\Gamma \tfrac{1}{2}(\underset{+}{\hat{N}_\nu} - \underset{-}{\hat{N}_\nu})\,\delta\langle u_3\rangle\, d\sigma \qquad & + \int_\Gamma \langle \hat{N}_\nu\rangle \tfrac{1}{2}\,\delta(\underset{+}{u_3} + \underset{-}{u_3})\, d\sigma
\end{aligned}
\tag{6.5}
$$

Integral theorem (C2.10) provides another transformation and that is

$$
\begin{aligned}
&-\int_A \epsilon^{\alpha\rho}\epsilon^{\beta\lambda}\hat{\Theta}_{\alpha\beta}\,\delta F|_{\rho\lambda}\, dA \\
&= -\int_A \epsilon^{\alpha\rho}\epsilon^{\beta\lambda}\hat{\Theta}_{\alpha\beta}|_{\rho\lambda}\,\delta F\, dA \qquad + \oint_C \hat{\Theta}_{tt}\,\delta\chi_t\, ds \\
&+\oint_C (-\hat{\Theta}_{\nu t,t} + \hat{\Theta}_{tt,\nu} - \hat{\Theta}_{\nu t,s})\,\delta F\, ds - [\,\hat{\Theta}_{\nu t}\,\delta F\,]_c
\end{aligned}
\tag{6.6}
$$

where the first of relations (5.2.10) was used. The paranthetical term of the last integral coincides with definition (5.3.9), such that

$$
\hat{k}_{nt} = -\hat{\Theta}_{\nu t,t} + \hat{\Theta}_{tt,\nu} - \hat{\Theta}_{\nu t,s}\,. \tag{6.7}
$$

Keeping in mind that discontinuities are admitted, one gets finally especially with (6.3)

$$
\begin{aligned}
&-\int_A \epsilon^{\alpha\rho}\epsilon^{\beta\lambda}\hat{\Theta}_{\alpha\beta}\,\delta F|_{\rho\lambda}\, dA \\
&= -\int_A \epsilon^{\alpha\rho}\epsilon^{\beta\lambda}\hat{\Theta}_{\alpha\beta}|_{\rho\lambda}\,\delta F\, dA \\
&+\oint_C [\,\hat{\Theta}_{tt}\,\delta\chi_t + \hat{k}_{nt}\,\delta F\,]\, ds - [\,\hat{\Theta}_{\nu t}\, F\,]_c \\
&+\int_\Gamma \tfrac{1}{2}(\underset{+}{\hat{\Theta}_{tt}} + \underset{-}{\hat{\Theta}_{tt}})\,\delta\langle \chi_t\rangle\, d\sigma \qquad +\int_\Gamma \langle \hat{\Theta}_{tt}\rangle \tfrac{1}{2}\,\delta(\underset{+}{\chi_t} - \underset{-}{\chi_t})\, d\sigma \\
&+\int_\Gamma \tfrac{1}{2}(\underset{+}{\hat{k}_{nt}} - \underset{-}{\hat{k}_{nt}})\,\delta\langle F\rangle\, d\sigma \qquad +\int_\Gamma \langle \hat{k}_{nt}\rangle \tfrac{1}{2}\,\delta(\underset{+}{F} + \underset{-}{F})\, d\sigma \\
&-\sum \tfrac{1}{2}(\underset{+}{\hat{\Theta}_{\nu t}} + \underset{-}{\hat{\Theta}_{\nu t}})\,\delta\langle F\rangle \qquad -\sum \langle \hat{\Theta}_{\nu t}\rangle \tfrac{1}{2}\,\delta(\underset{+}{F} + \underset{-}{F})\,.
\end{aligned}
\tag{6.8}
$$

It is not necessary, to examine also the bending part of the functional, because its variational statement is the same as in the variation of Hellinger-Reissner's principle (s. 4.11). For that reason only the membrane part of the Hu-Washizu functional is interesting and shall be varied.

$$\delta I_H = \textit{VARIATION OF THE BENDING PART} +$$

$$- \int_A \left(\frac{\partial W_M}{\partial \gamma_{\alpha\beta}} - N^{\alpha\beta} \right) \delta \gamma_{\alpha\beta} \, dA \qquad \Big\} \text{ constit. equations}$$

$$- \int_A \left(\gamma_{\alpha\beta} - \underline{\hat{\theta}_{\alpha\beta}} - \tfrac{1}{2} u_{3,\alpha} u_{3,\beta} \right) \delta N^{\alpha\beta} dA \qquad \Bigg\} \text{ strain-displ.-relations}$$

$$+ \int_A \left(N^{\alpha\beta} - \varepsilon^{\alpha\varrho} \varepsilon^{\beta\lambda} F|_{\varrho\lambda} \right) \delta \hat{\theta}_{\alpha\beta} dA \qquad \Bigg\} \text{ stress-stressf.-relations}$$

$$- \int_A \varepsilon^{\alpha\varrho} \varepsilon^{\beta\lambda} \hat{\theta}_{\alpha\beta}|_{\varrho\lambda} \, \delta F \, dA \qquad \Big\} \text{ compat. cond.}$$

$$+ \int_{C_2} (F^* - F) \delta \hat{R} \, ds + \int_{C_2} (X_t^* - X_t) \delta \hat{D} \, ds - [(F^* - F) \delta \hat{R}_t]_{C_2} \qquad \Bigg\} \text{ statical boundary cond.}$$

$$- \int_{C_1} (k_{nt}^* - \hat{k}_{nt}) \delta F \, ds - \int_{C_1} (D_t^* - \hat{\theta}_{tt}) \delta X \, ds + [(D_v^* - \hat{\theta}_{vt}) \delta F]_{C_1} \qquad \Bigg\} \text{ geom. boundary cond.}$$

$$- \int_\Gamma \langle F \rangle \delta \hat{f} \, d\sigma \qquad - \int_\Gamma \langle X_t \rangle \delta \hat{s} \, d\sigma \qquad + \sum_G \langle F_G \rangle \delta \hat{S}_G \qquad \Bigg\} \text{ statical interel. cond.}$$

$$\left. \begin{array}{l} - \int_\Gamma [k_{nt}^* - \langle k_{nt} \rangle] \tfrac{1}{2} \delta (\underset{+}{F} + \underset{-}{F}) d\sigma - \int_\Gamma [D_t^* - \langle \theta_{tt} \rangle] \tfrac{1}{2} \delta (\underset{+}{X_t} - \underset{-}{X_t}) d\sigma \\ \qquad\qquad\qquad - \sum \langle \hat{\theta}_{tv} \rangle \tfrac{1}{2} \delta (\underset{+}{F} - \underset{-}{F}) \end{array} \right\} \text{ geom. interel. cond.}$$

$$\left. \begin{array}{l} + \int_{C_2} (\underline{\hat{R}} - \hat{k}_{nt}) \delta F ds - \int_{C_2} (\underline{\hat{D}} - \hat{\theta}_{tt}) \delta X_t ds + [(\underline{\hat{R}_t} - \hat{\theta}_{vt}) \delta F]_{C_2} \\ - \int_\Gamma [\underline{\hat{f}} - \tfrac{1}{2} (\underset{+}{\hat{k}_{nt}} - \underset{-}{\hat{k}_{nt}})] \delta \langle F \rangle d\sigma \\ - \int_\Gamma [\underline{\hat{s}} - \tfrac{1}{2} (\underset{+}{\hat{\theta}_{tt}} + \underset{-}{\hat{\theta}_{tt}})] \delta \langle X_t \rangle d\sigma - \sum [\underline{\hat{S}_G} - \tfrac{1}{2} (\underset{+}{\hat{\theta}_{tv}} - \underset{-}{\hat{\theta}_{tv}})] \delta \langle F \rangle \end{array} \right\} \text{ LAGRANGE FACTORS}$$

(6.9)

Due to the expositions of chapter 3 the linear independence of all varied quantities is guaranteed. Consequently, the *Euler-Lagrange-equations*, the *transitional* and *boundary conditions* for the membrane part are obtained. Out of this, the physical meaning of all *Lagrange multipliers* is clarified.

The second integral of (6.9) shows the strain-displacement relation, which serves as equation of definition for the linear strain tensor having been a Lagrange factor.

$$\Theta_{\alpha\beta} = \gamma_{\alpha\beta} - \tfrac{1}{2} u_{3,\alpha} u_{3,\beta} \tag{6.10}$$

All quantities defined by $\Theta_{\alpha\beta}$ must be expressed by (6.10). Especially, this applies for the compatibility equation (C2.13).

$$\epsilon^{\alpha\rho} \epsilon^{\beta\lambda} \Theta_{\alpha\beta}/_{\rho\lambda} = 0 \tag{6.11}$$

In (6.11) reference is taken to (6.10).

$$\epsilon^{\alpha\rho} \epsilon^{\beta\lambda} (\gamma_{\alpha\beta} - \tfrac{1}{2} u_{3,\alpha} u_{3,\beta})/_{\rho\lambda} = 0 \tag{6.12}$$

This equation can be transformed. At first, it follows

$$\begin{aligned} & \epsilon^{\alpha\rho} \epsilon^{\beta\lambda} (u_{3,\alpha} u_{3,\beta})/_{\rho} \\ = \; & \epsilon^{\alpha\rho} \epsilon^{\beta\lambda} (u_3/_{\alpha\rho}\, u_{3,\beta} + u_{3,\alpha}\, u_3/_{\beta\rho}) \\ = \; & \epsilon^{\alpha\rho} \epsilon^{\beta\lambda} u_{3,\alpha}\, u_3/_{\beta\rho} \, . \end{aligned}$$

After that, subsequent differentiation yields by paying attention to the fact of interchangeable sequence of derivatives

$$\begin{aligned} & \epsilon^{\alpha\rho} \epsilon^{\beta\lambda} (u_{3,\alpha}\, u_3/_{\beta\rho})/_{\lambda} \\ = \; & \epsilon^{\alpha\rho} \epsilon^{\beta\lambda} (u_3/_{\alpha\lambda}\, u_3/_{\beta\rho} + u_{3,\alpha}\, u_3/_{\beta\rho\lambda}) \\ = \; & \epsilon^{\alpha\rho} \epsilon^{\beta\lambda} u_3/_{\alpha\lambda}\, u_3/_{\beta\rho} \\ = \; & \delta^{\alpha\rho}_{\beta\lambda}\, u_3/^{\lambda}_{\alpha}\, u_3/^{\beta}_{\rho} = -2 \det(u_3/^{\beta}_{\alpha}) = -2K \, . \end{aligned}$$

Alternatively the compatibility equation (6.11) gains the equivalent formulation

$$\epsilon^{\alpha\rho} \epsilon^{\beta\lambda} \gamma_{\alpha\beta}/_{\rho\lambda} + K = 0 \tag{6.13}$$

Herein, according to (A3.13), the quantity

$$K = \det(u_3|_\alpha^\beta) = \det(\chi_\alpha^\beta)$$

is the *Gaußian curvature* of the plate.

It still remains, to study the case, if the linear strain tensor is subject to variation instead of the nonlinear and the functional (5.4.1) is put as a foundation. In this case, only the first and the fifth integral of the functional (6.1) will change.

$$I_H = \int_A W_M(\theta_{\alpha\beta} + \tfrac{1}{2}u_{3,\alpha}u_{3,\beta})\,dA - \cdots$$

$$\cdots - \int_A N^{\alpha\beta}\theta_{\alpha\beta}\,dA - \cdots \tag{6.14}$$

The first variation of the function of energy density is now

$$\delta W_M = \frac{\partial W_M}{\partial \theta_{\alpha\beta}}\,\delta\theta_{\alpha\beta} + \frac{\partial W_M}{\partial u_{3,\alpha}}\,\delta u_{3,\alpha}$$

where
$$\frac{\partial W_M}{\partial \theta_{\alpha\beta}} = \frac{\partial W_M}{\partial \gamma_{\alpha\beta}}$$
$$\frac{\partial W_M}{\partial u_{3,\alpha}} = \frac{\partial W_M}{\partial \gamma_{\alpha\beta}}\,u_{3,\beta} = N^\alpha$$

easily can be recognized. In the last expression use has been made of definition (2.16).

Hence, it follows

$$\int_A \delta W_M\,dA = \int_A \frac{\partial W_M}{\partial \gamma_{\alpha\beta}}\,\delta\theta^{\alpha\beta}\,dA + \int_A N^\alpha\,\delta u_{3,\alpha}\,dA\,.$$

Obviously, in consequence of these considerations the variational statement of (6.9) is not affected.

7. REDUCTION OF THE FUNCTIONALS

7.1. Some transformations by means of partial integration

The variational principles (4.1), (6.1) and (6.14) in the present form still exhibit a decisive disadvantage. High degrees of discontinuities of the deflection u_3 and of the Airy stress function F cause a lot of numerical work at interelement boundaries.
This defficency can be removed, if repeated integration by parts is performed in all three functionals. In [14] such a procedure was described for the first time in linear plate theory.
The functional of Hellinger-Reissner and of Hu-Washizu contain the following two integrals:

$$-\int_A M^{\alpha\beta} u_3|_{\alpha\beta}\, dA - \int_\Gamma \langle \beta_\nu \rangle\, \hat{m}\, d\sigma \tag{7.1.1}$$

where due to (4.11) $\hat{m}$ is a Lagrange factor defined by

$$\hat{m} = \frac{1}{2}\left(\underset{+}{M}_{\nu\nu} + \underset{-}{M}_{\nu\nu}\right).$$

After integration by parts of the first integral one receives

$$\begin{aligned}
&-\int_A M^{\alpha\beta} u_3|_{\alpha\beta}\, dA \qquad - \int_\Gamma \langle \beta_\nu \rangle \frac{1}{2}\left(\underset{+}{M}_{\nu\nu} + \underset{-}{M}_{\nu\nu}\right) d\sigma \\
&= \int_A M^{\alpha\beta}|_\beta\, u_{3,\alpha}\, dA \qquad + \oint_C \left(M_{\nu\nu}\beta_\nu + M_{t\nu}\beta_t\right) ds \\
&\qquad\qquad + \int_\Gamma \langle M_{\nu\nu} \rangle \frac{1}{2}\left(\underset{+}{\beta}_\nu - \underset{-}{\beta}_\nu\right) d\sigma \\
&+ \int_\Gamma \frac{1}{2}\left(\underset{+}{M}_{t\nu} + \underset{-}{M}_{t\nu}\right) \langle \beta_t \rangle\, d\sigma \qquad + \int_\Gamma \langle M_{t\nu} \rangle \frac{1}{2}\left(\underset{+}{\beta}_t - \underset{-}{\beta}_t\right) d\sigma .
\end{aligned} \tag{7.1.2}$$

That means in case of continuity of the bending moment $M_{\nu\nu}$ that the deflection u_3 of the plate needs not to satisfy the transitional condition

$$\langle \beta_\nu \rangle = 0 .$$

Analogously, the functionals (6.1) and (6.14) respectively must be treated with respect to the Airy stress function F. The corresponding integrals are of the form

$$-\int_A \epsilon^{\kappa\rho}\epsilon^{\beta\lambda}\Theta_{\kappa\beta}\,F|_{\rho\lambda}\,dA \;-\; \int_\Gamma \langle \chi_t \rangle\,\hat{s}\,d\sigma \qquad (7.1.3)$$

with the Lagrangefactor

$$\hat{s} = \tfrac{1}{2}(\underset{+}{\Theta_{tt}} + \underset{-}{\Theta_{tt}}).$$

The surface integral shall be integrated by parts. For the sake of simplicity, the tensor $T^{\rho\lambda}$ may be defined for the present.

$$T^{\rho\lambda} = \epsilon^{\alpha\rho}\epsilon^{\beta\lambda}\Theta_{\alpha\beta}$$

Presuming continuity of the field quantities Gauß' theorem provides at first with (5.2.10) the transformation

$$\begin{aligned} &-\int_A T^{\rho\lambda} F|_{\rho\lambda}\,dA \\ &= \int_A T^{\rho\lambda}|_\lambda\, F_{,\rho}\,dA \qquad - \oint_C T^{\rho\lambda} F_{,\rho}\,\nu_\lambda\,ds \\ &= \int_A \epsilon^{\kappa\rho}\epsilon^{\beta\lambda}\Theta_{\kappa\beta} F_{,\rho}\,dA \qquad + \oint_C (\Theta_{tt}\chi_t + \Theta_{\nu t}\chi_\nu)\,ds. \end{aligned} \qquad (7.1.4)$$

If discontinuities are included at interelement boundaries, from (7.1.4) follows entirely for (7.1.3)

$$\begin{aligned} &-\int_A \epsilon^{\alpha\rho}\epsilon^{\beta\lambda}\Theta_{\alpha\beta} F|_{\rho\lambda}\,dA \qquad -\int_\Gamma \langle\chi_t\rangle \tfrac{1}{2}(\underset{+}{\Theta_{tt}} + \underset{-}{\Theta_{tt}})\,d\sigma \\ &= \int_A \epsilon^{\alpha\rho}\epsilon^{\beta\lambda}\Theta_{\alpha\beta}|_\lambda F_{,\rho}\,dA \qquad + \oint_C (\Theta_{tt}\chi_t + \Theta_{\nu t}\chi_\nu)\,ds \\ &+ \int_\Gamma \langle\Theta_{tt}\rangle \tfrac{1}{2}(\underset{+}{\chi_t} - \underset{-}{\chi_t})\,d\sigma \\ &+ \int_\Gamma \langle\chi_\nu\rangle \tfrac{1}{2}(\underset{+}{\Theta_{\nu t}} + \underset{-}{\Theta_{\nu t}})\,d\sigma + \int_\Gamma \langle\Theta_{\nu t}\rangle \tfrac{1}{2}(\underset{+}{\chi_\nu} - \underset{-}{\chi_\nu})\,d\sigma. \end{aligned} \qquad (7.1.5)$$

The significance of this transformation is evident in the same sense as it was for the deflection u_3 in (7.1.2). Independent of continuity or disconitnuity of the normal derivative $\chi_t = -F_{,\nu}$ of the stress function the line integral vanishes, provided the linear stretching quantity θ_{tt} is continuous.

Before presenting the final form of the functionals another integral transformation may be regarded for the Reissner functional. Recalling the definition of the Lagrange factors $\hat{s}_t$ and $\hat{s}_\nu$ from (4.11), it follows

$$\int_A N^{\alpha\beta}\tfrac{1}{2}(u_\alpha|_\beta + u_\beta|_\alpha)\,dA \;-\int_\Gamma \langle u_\nu\rangle \hat{s}_\nu\,d\sigma \;-\int_\Gamma \langle u_t\rangle \hat{s}_t\,d\sigma$$

$$= -\int_A N^{\alpha\beta}|_\beta\, u_\alpha\,dA \;+\oint_C (N_{\nu\nu}u_\nu + N_{t\nu}u_t)\,ds$$

$$+\int_\Gamma \langle N_{\nu\nu}\rangle \tfrac{1}{2}(\overset{+}{u}_\nu - \overset{-}{u}_\nu)\,d\sigma \;+\int_\Gamma \langle N_{t\nu}\rangle \tfrac{1}{2}(\overset{+}{u}_t - \overset{-}{u}_t)\,d\sigma\,. \qquad (7.1.6)$$

7.2. The reduced form of the functionals

The relations (7.1.2), (7.1.5) and (7.1.6) specify the functionals to their final form, which is determined for numerical calculations.

The modified Reissner functional:

$$I_R = -\int_A W_c(N^{\alpha\beta}, M^{\alpha\beta})\,dA \;-\int_A p^* u_3\,dA$$

$$-\int_A N^{\alpha\beta}|_\beta\, u_\alpha\,dA \;+\int_A \tfrac{1}{2}N^{\alpha\beta}u_{3,\alpha}u_{3,\beta}\,dA$$

$$+\int_A M^{\alpha\beta}|_\beta\, u_{3,\alpha}\,dA$$

$$+\int_{C_1}(u_t^* N_{t\nu} + u_\nu^* N_{\nu\nu})\,ds \;+\oint_C M_{t\nu}\beta_t\,ds$$

$$+\int_{C_1}(u_3^* - u_3)(P_{\nu\nu} + N_\nu)\,ds$$

$$+\left[(u_3^* - u_3)M_{t\nu}\right]_{C_1} \;+\int_{C_1}\beta_\nu^* M_{\nu\nu}\,ds$$

$$-\int_{C_2}(P_{\nu\nu}^{*} - N_{\nu\nu})\,u_\nu\,ds \quad -\int_{C_2}(P_{t\nu}^{*} - N_{t\nu})\,u_t\,ds$$

$$-\int_{C_2}(K_\nu^{*} - M_{\nu\nu})\,\beta_\nu\,ds \quad -\int_{C_2} P_{n\nu}^{*}\,u_3\,ds$$

$$-[\,K_t^{*}\,u_3\,]_{C_2}$$

$$-\int_{\Gamma} P_{n\nu}^{*}\,\tfrac{\mp}{2}(\underset{+}{u_3} + \underset{-}{u_3})\,d\sigma \quad -\int_{\Gamma} K_\nu^{*}\,\tfrac{\mp}{2}(\underset{+}{\beta_\nu} - \underset{-}{\beta_\nu})\,d\sigma$$

$$+\int_{\Gamma}\langle N_{\nu\nu}\rangle\,\tfrac{\mp}{2}(\underset{+}{u_\nu} - \underset{-}{u_\nu})\,d\sigma \quad +\int_{\Gamma}\langle N_{t\nu}\rangle\,\tfrac{\mp}{2}(\underset{+}{u_t} - \underset{-}{u_t})\,d\sigma$$

$$+\int_{\Gamma}\langle M_{\nu\nu}\rangle\,\tfrac{\mp}{2}(\underset{+}{\beta_\nu} - \underset{-}{\beta_\nu})\,d\sigma$$

$$+\int_{\Gamma}\langle M_{t\nu}\rangle\,\tfrac{\mp}{2}(\underset{+}{\beta_t} - \underset{-}{\beta_t})\,d\sigma \quad +\int_{\Gamma}\langle \beta_t\rangle\,\tfrac{\mp}{2}(\underset{+}{M_{t\nu}} + \underset{-}{M_{t\nu}})\,d\sigma$$

$$-\int_{\Gamma}\tfrac{\mp}{2}\left[(\underset{+}{P_{n\nu}} + \underset{+}{N_\nu}) - (\underset{-}{P_{n\nu}} + \underset{-}{N_\nu})\right]\langle u_3\rangle\,d\sigma$$

$$\sum \langle u_3\rangle\,\tfrac{\mp}{2}(\underset{+}{M_{t\nu}} - \underset{-}{M_{t\nu}}) \qquad (7.2.1)$$

The modified Hu-Washizu functional:

$$I_H = \int_A [W_H(\gamma_{\alpha\beta}) - W_{CB}(M^{\alpha\beta})]\,dA \quad -\int_A p^{*}u_3\,dA$$

$$+\int_A M^{\alpha\beta}|_\beta\,u_{3,\alpha}\,dA \quad +\int_A \epsilon^{\alpha\varrho}\epsilon^{\beta\lambda}\Theta_{\alpha\beta}|_\lambda\,F_{,\varrho}\,dA$$

$$+\oint_C M_{t\nu}\,\beta_t\,ds \quad +\oint_C \Theta_{\nu t}\,\chi_\nu\,ds$$

$$+\int_{C_1}(u_3^{*} - u_3)(P_{n\nu} + N_\nu)\,ds \quad +\int_{C_1}\beta_\nu^{*}\,M_{\nu\nu}\,ds$$

$$-\int_{C_1} k_{nt}^{*}\,F\,ds \quad -\int_{C_1}(D_t^{*} - \Theta_{tt})\,\chi_t\,ds$$

$$
\begin{aligned}
&-\left[(F^{*}-F)\,\Theta_{\nu t}\right]_{C_2} \qquad -\left[K_t^{*}\,u_3\right]_{C_2} \\
&-\int_\Gamma p_{n\nu}^{*}\,\tfrac{1}{2}(\underset{+}{u_3}+\underset{-}{u_3})\,d\sigma \qquad -\int_\Gamma K_\nu^{*}\,\tfrac{1}{2}(\underset{+}{\beta_\nu}-\underset{-}{\beta_\nu})\,d\sigma \\
&-\int_\Gamma D_t^{*}\,\tfrac{1}{2}(\underset{+}{\chi_t}-\underset{-}{\chi_t})\,d\sigma \qquad -\int_\Gamma k_{nt}^{*}\,\tfrac{1}{2}(\underset{+}{F}+\underset{-}{F})\,d\sigma \\
&+\int_\Gamma \langle M_{\nu\nu}\rangle\,\tfrac{1}{2}(\underset{+}{\beta_\nu}-\underset{-}{\beta_\nu})\,d\sigma \qquad +\int_\Gamma \langle \Theta_{tt}\rangle\,\tfrac{1}{2}(\underset{+}{\chi_t}-\underset{-}{\chi_t})\,d\sigma \\
&+\int_\Gamma \langle M_{t\nu}\rangle\,\tfrac{1}{2}(\underset{+}{\beta_t}-\underset{-}{\beta_t})\,d\sigma \qquad +\int_\Gamma \langle \Theta_{\nu t}\rangle\,\tfrac{1}{2}(\underset{+}{\chi_\nu}-\underset{-}{\chi_\nu})\,d\sigma \\
&+\int_\Gamma \langle \beta_t\rangle\,\tfrac{1}{2}(\underset{+}{M_{t\nu}}+\underset{-}{M_{t\nu}})\,d\sigma \qquad +\int_\Gamma \langle \chi_\nu\rangle\,\tfrac{1}{2}(\underset{+}{\Theta_{\nu t}}+\underset{-}{\Theta_{\nu t}})\,d\sigma \\
&-\int_\Gamma \langle u_3\rangle\,\tfrac{1}{2}\left[(\underset{+}{p_{n\nu}}+\underset{-}{N_\nu})-(\underset{+}{p_{n\nu}}+\underset{-}{N_\nu})\right]d\sigma \\
&-\int_\Gamma \langle F\rangle\,\tfrac{1}{2}(\underset{+}{k_{nt}}-\underset{-}{k_{nt}})\,d\sigma \\
&-\sum \langle u_3\rangle\,\tfrac{1}{2}(\underset{+}{M_{t\nu}}-\underset{-}{M_{t\nu}}) + \sum \langle F\rangle\,\tfrac{1}{2}(\underset{+}{\Theta_{t\nu}}-\underset{-}{\Theta_{t\nu}})
\end{aligned}
\tag{7.2.2}
$$

The side condition

$$\gamma_{\alpha\beta} - \Theta_{\alpha\beta} - \tfrac{1}{2}\,u_{3,\alpha}\,u_{3,\beta} = 0$$

offers the possibility, to treat the Hu-Washizu functional with $\gamma_{\alpha\beta}$ or $\Theta_{\alpha\beta}$ as independent variables in choice.

8. NUMERICAL APPLICATIONS

8.1. Prelaminaries

Referring mainly to the simply supported plate with unmovable edges the functionals (7.2.1) and (7.2.2) are proved with respect to their numerical applicability.

Only a constant load P^* exists, measured per unit area of the undeformed configuration. Now, by introduction of dimensionless quantities the numerical treatment of all examples is performed with respect to dimensionless coordinates x_1, x_2. Original quantities are supplied by a dash and are referred to the original system $\bar{x}_1$, $\bar{x}_2$.

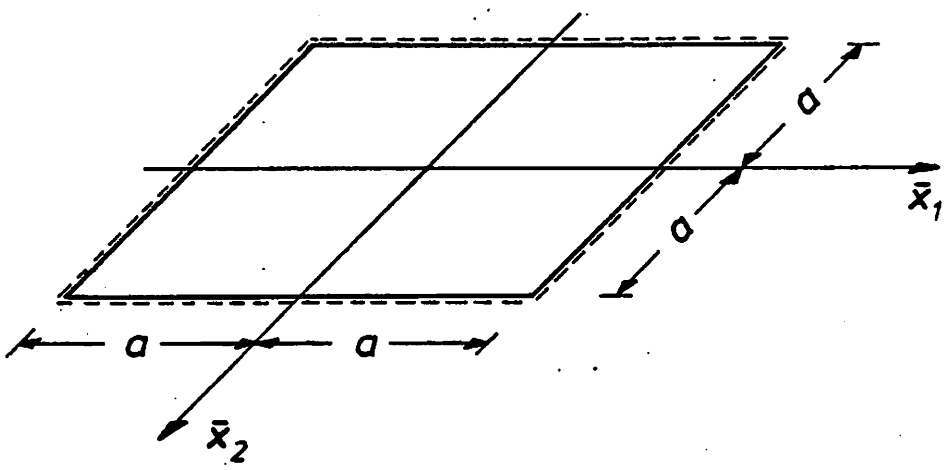

Fig. 3

The relations between original and normalized field quantities are as follows:

$$\left.\begin{array}{llll}
\bar{M}_{\kappa\beta} = \dfrac{Eh^3}{a^2} M_{\kappa\beta} & , & \bar{N}_{\kappa\beta} = \dfrac{Eh^3}{a^2} N_{\alpha\beta} \\
\bar{\gamma}_{\kappa\beta} = \dfrac{h^2}{a^2} \gamma_{\kappa\beta} & , & \bar{\Theta}_{\kappa\beta} = \dfrac{h^2}{a^2} \Theta_{\kappa\beta} \\
\bar{u}_3 = h u_3 & , & \bar{u}_\kappa = \dfrac{h^2}{a} u_\kappa \\
\bar{F} = \dfrac{Eh^3}{a^2} F & , & \bar{p}^* = \dfrac{Eh^4}{a^4} p^*
\end{array}\right\} \qquad (8.1.1)$$

The application of normalized variables proves to be very useful for programming and shows that numerical values of all field quantities have no big differences to each other. The dimension of the functionals always

amounts to Eh^5/a^2.
Following the modified functionals (7.2.1) and (7.2.2), numerical examples will be presented for different kind of element shape functions. Either quadratic or triangular elements are chosen as type of elements. Because of the symmetry of the problems, it suffices to refer all subsequent considerations constantly to the first quadrant of the plate. At last,numerical calcualtions result in solving nonlinear equations by the iterative scheme of Newton-Raphson.

8.2. Numerical examples for the simply supported plate

The boundary conditions for the Reissner functional read

$$\left.\begin{array}{lll} \text{on } C_1: & u_3 = 0\,, & u_\nu = 0 \\ & & u_t = 0 \\ \text{on } C_2: & & M_{\nu\nu} = 0 \end{array}\right\} \qquad (8.2.1)$$

On the other hand Washizu's functional is related to the boundary conditions

$$\left.\begin{array}{lll} \text{on } C_1: & u_3 = 0\,, & k_{\nu t} = 0 \\ & & \gamma_{tt} = 0 \\ & & \theta_{tt} = 0 \\ \text{on } C_2: & & M_{\nu\nu} = 0 \end{array}\right\} \qquad (8.2.2)$$

EXAMPLE 1:	*Modified Hellinger-Reissner funktional*
Type of element:	*quadratic elements*
Shape functions:	
$M_{\alpha\beta}$, $N_{\alpha\beta}$, u_3, u_α	*linear*
Number of unknowns:	183

From functional (7.2.1) follows that no jump terms appear.

$$
\begin{aligned}
I_{R1} = &-\int_A W_c(N_{\alpha\beta}, M_{\alpha\beta})\, dA &&- \int_A p^* u_3\, dA \\
&-\int_A N_{\alpha\beta,\beta}\, u_\alpha\, dA &&+ \int_A \tfrac{\xi}{2} N_{\alpha\beta}\, u_{3,\alpha}\, u_{3,\beta}\, dA \\
&+\int_A M_{\alpha\beta,\beta}\, u_{3,\alpha}\, dA
\end{aligned}
\tag{8.2.3}
$$

Numerical results are shown in figure 6 to 11.

EXAMPLE 2: *Modified Hellinger-Reissner functional*

Type of element: *quadratic elements*

Shape functions:

$M_{\alpha\beta}$, u_3, u_α	*linear*
$N_{\alpha\beta}$	*constant*

Number of unknowns: 171

By choice of constant membrane stresses for each element additional jump terms arise.

$$
\begin{aligned}
I_{R2} = &-\int_A W_c(N_{\alpha\beta}, M_{\alpha\beta})\, dA &&- \int_A p^* u_3\, dA \\
&+\int_A M_{\alpha\beta,\beta}\, u_{3,\alpha}\, dA &&+ \int_A \tfrac{\xi}{2} N_{\alpha\beta}\, u_{3,\alpha}\, u_{3,\beta}\, dA \\
&+\int_\Gamma \langle N_{\nu\nu} \rangle\, u_\nu\, d\sigma &&+ \int_\Gamma \langle N_{t\nu} \rangle\, u_t\, d\sigma
\end{aligned}
\tag{8.2.4}
$$

Figure 12 and 13 show a graphical representation of membrane stresses. The other results vary scarcely from those of the previous example. Figure 14 gives a graphical demonstration of values of I_{R2} depending on the vertical load p^*, where the free parameter n stands for the number of elements of the whole plate.

EXAMPLE 3:   *Modified Hellinger-Reissner functional*

Type of element: *quadratic elements*

Shape functions:

M_{11}, N_{11}	*linear in* x_1*-direction* *constant in* x_2*-direction*
M_{22}, N_{22}	*linear in* x_2*-direction* *constant in* x_1*-direction*
u_α, u_3	*linear*
N_{12}, M_{12}	*constant*

Number of unknowns: 171

$$I_{R3} = I_{R1} + \int_\Gamma \langle N_{t\nu} \rangle u_t \, d\sigma \qquad + \int_\Gamma \langle M_{t\nu} \rangle \beta_t \, d\sigma \tag{8.2.5}$$

With the exception of u_α, u_3 and N_{12} the results are plotted in figure 15 to 17.

EXAMPLE 4:   *Modified Hellinger-Reissner functional*

Type of elements:

Shape functions:

$M_{\nu\nu}$, $N_{\nu\nu}$	*constant on* Γ
u_3, u_α	*linear*

Number of unknowns: 171

The simple form of the shape functions reduces the functional (7.2.1) by the third and the fifth surface integral.

$$\begin{aligned} I_{R4} = -\int_A W_c(N_{\alpha\beta}, M_{\alpha\beta}) \, dA \qquad & - \int_A p^* u_3 \, dA \\ & + \int_A \tfrac{1}{2} N_{\alpha\beta} u_{3,\alpha} u_{3,\beta} \, dA \\ + \int_\Gamma \langle N_{t\nu} \rangle u_t \, d\sigma \qquad & \int_\Gamma \langle M_{t\nu} \rangle \beta_t \, d\sigma \end{aligned} \tag{8.2.6}$$

The trial function $M_{\nu\nu}$ = const is based on a concept of Hermann [13] and is known only in linear plate theory. Now, in von Kármán's plate theory a similar element function is chosen for membrane stresses. By this, along the sides of any triangular plate-element constant but continuous bending moments $M_{\nu\nu}$ and constant but continuous normal stresses $N_{\nu\nu}$ are given.

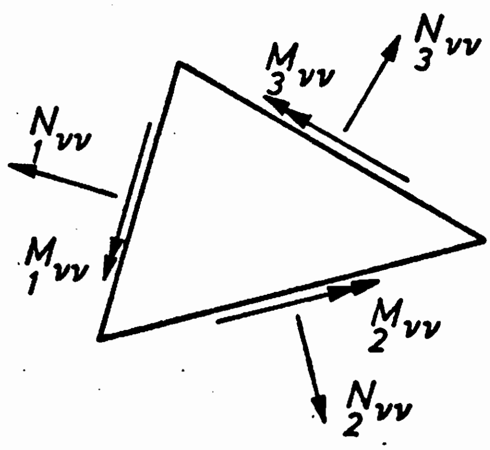

Fig. 4

For each side of the triangle the following relations hold (i = 1,2,3):

$$\underset{i}{M}_{\nu\nu} = M_{\alpha\beta}\, \underset{i}{\nu}_{\alpha}\, \underset{i}{\nu}_{\beta} \tag{8.2.7}$$

$$\underset{i}{N}_{\nu\nu} = N_{\alpha\beta}\, \underset{i}{\nu}_{\alpha}\, \underset{i}{\nu}_{\beta} \tag{8.2.8}$$

$$M_{\alpha\beta} = \sum_{i} \underset{i}{c}_{\alpha\beta}\, \underset{i}{M}_{\nu\nu} \tag{8.2.9}$$

$$N_{\alpha\beta} = \sum_{i} \underset{i}{c}_{\alpha\beta}\, \underset{i}{M}_{\nu\nu} \tag{8.2.10}$$

The tensors $M_{\alpha\beta}$ and $N_{\alpha\beta}$ are solutions of the equations (8.2.7) and (8.2.8), where the coefficients $c_{i\alpha\beta}$ are symmetrical. Finally, the twisting moment $M_{t\nu}$ and the shear stress vector $N_{t\nu}$ are specified on each side by

$$\underset{i}{M}_{t\nu} = M_{\alpha\beta}\, \underset{i}{t}_{\alpha}\, \underset{i}{\nu}_{\beta} \tag{8.2.11}$$

$$\underset{i}{N}_{t\nu} = N_{\alpha\beta}\, \underset{i}{t}_{\alpha}\, \underset{i}{\nu}_{\beta}\,. \tag{8.2.12}$$

With relations (8.2.7) to (8.2.12) the functional (8.2.6) has been optimized. The results are to be seen in figure 18 to 21.

EXAMPLE 5: *Modified Hu-Washizu functional*

type of elements: *quadratic elements*

Shape functions:

$M_{\alpha\beta}$, $\Theta_{\alpha\beta}$, u_3, F *linear*

Number of unknowns: 173

Owing to the continuity of the shape functions all jump terms disappear in functional (7.2.2)

$$
\begin{aligned}
I_{H1} = {} & \int_A W_H(\Theta_{\alpha\beta} + \tfrac{1}{2} u_{3,\alpha} u_{3,\beta})\, dA && - \int_A W_{CB}(M_{\alpha\beta})\, dA \\
& + \int_A M_{\alpha\beta,\beta}\, u_{3,\alpha}\, dA && - \int_A p^* u_3\, dA \\
& + \int_A \epsilon_{\alpha\varrho}\, \epsilon_{\beta\lambda}\, \Theta_{\alpha\beta,\lambda}\, F_{,\varrho}\, dA && + \oint_C \Theta_{\nu\tau}\, \chi_\tau\, ds
\end{aligned}
\qquad (8.2.13)
$$

A graphical representation of results can be found in figure 22 to 24.

EXAMPLE 6: *Modified Hu-Washizu functional*

Type of elements: *quadratic elements*

Shape functions:

$M_{\alpha\beta}$, $\gamma_{\alpha\beta}$, u_3, F *linear*

Number of unknowns: 173

The continuity of the shape functions leads to the fact that the following relations hold on the boundaries.

$$
\left.
\begin{aligned}
&\text{On } C: && u_{3,t} = 0 \\
&\text{On } \Gamma: && \langle u_{3,t} \rangle = 0 \\
& && \langle \gamma_{\nu t} \rangle = 0
\end{aligned}
\right\}
\qquad (8.2.14)
$$

Because of the discontinuity of $\beta_\nu = -u_{3,\nu}$ only one jump term remains in the functional (7.2.2).

$$
\begin{aligned}
I_{H2} = {} & \int_A W_H(\gamma_{\alpha\beta})\, dA && - \int_A W_{CB}(M_{\alpha\beta})\, dA \\
& + \int_A M_{\alpha\beta,\beta}\, u_{3,\alpha}\, dA && - \int_A p^* u_3\, dA
\end{aligned}
$$

$$+\int_A \epsilon_{\alpha\varrho}\,\epsilon_{\beta\lambda}\,(\gamma_{\alpha\beta} - \tfrac{1}{2}u_{3,\alpha}u_{3,\beta})_{,\lambda}\,F_{,\varrho}\,dA$$

$$+\oint_C \gamma_{\nu t}\,X_\nu\,ds \qquad -\int_\Gamma \langle\beta_\nu\rangle\,\tfrac{1}{2}\beta_t\,X_t\,d\sigma \tag{8.2.15}$$

The results of interest are shown in figure 25 and 26.
In this case too, a demonstration of the remaining results is not required, because numerical divergence compared to previous results is not worth mentioning. It is very interesting to compare figure 25 and 26 with figure 22 and 23. Approximately, the functional satisfies the transitional condition $u_{3,1} = 0$ along the x_2-axis. This means on $\Gamma = x_2$

$$\gamma_{11} = \Theta_{11} + \tfrac{1}{2}(u_{3,1})^2 \simeq \Theta_{11}\ .$$

Moreover, $u_{3,t} = 0$ is a property of u_3 on the boundary C. Hence, it follows on C:

$$\gamma_{\nu t} = \Theta_{\nu t} + \tfrac{1}{2}u_{3,\nu}u_{3,t} = \Theta_{\nu t}$$

EXAMPLE 7:	*Modified Hu-Washizu functional*
Type of elements:	*triangular elements*
Shape functions:	
$M_{\nu\nu}$, γ_{tt}	*constant on* Γ
u_3, F	*linear*
Number of unknowns:	162

The choice of these shape functions implies that two surface integrals drop out in (7.2.2). Paying attention to

$$u_{3,t} = 0 \qquad \text{on C}$$

$$\langle u_{3,t}\rangle = 0 \qquad \text{on } \Gamma$$

the resulting functional is of the form:

$$I_{H3} = \int_A [W_M(\gamma_{\alpha\beta}) - W_{CB}(M_{\alpha\beta})]\,dA \quad -\int_A p^* u_3\,dA$$

$$+\oint_C \gamma_{\nu t}\,X_\nu\,ds \qquad +\int_\Gamma \langle\gamma_{\nu t}\rangle\,X_\nu\,d\sigma$$

$$-\int_\Gamma \tfrac{1}{2}\langle\beta_\nu\rangle\beta_t\,X_t\,d\sigma \quad +\int_\Gamma \langle M_{t\nu}\rangle\,\beta_t\,d\sigma \tag{8.2.16}$$

All facts concerning $M_{\nu\nu}$ are treated already in example 4. The considerations for γ_{tt} are quite the same.

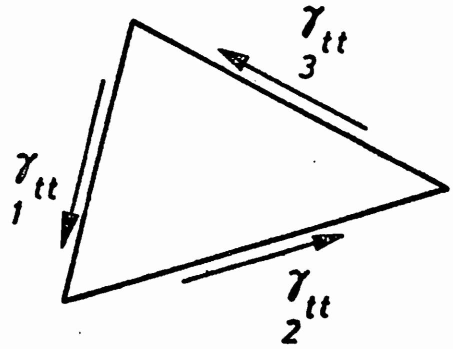

Fig. 5

On each side of the triangle the relations

$$\underset{i}{\gamma}_{tt} = \gamma_{\alpha\beta}\, \underset{i}{t}_{\alpha}\, \underset{i}{t}_{\beta} \tag{8.2.17}$$

are evident.

The solution for the inverse of equation (8.2.17) is

$$\gamma_{\alpha\beta} = \sum_{i} \underset{i}{d}_{\alpha\beta}\, \underset{i}{\gamma}_{tt} \,. \tag{8.2.18}$$

In consequence of the symmetry of the strain tensor the factors $d_{i\alpha\beta}$ are symmetrical too.

The tensor $\gamma_{\alpha\beta}$ defines subsequently all shear strains on the boundaries of the triangle.

$$\underset{i}{\gamma}_{\nu t} = \gamma_{\alpha\beta}\, \underset{i}{\nu}_{\alpha}\, \underset{i}{t}_{\beta} \tag{8.2.19}$$

Only results of interest are represented in figure 27 and 28.

__EXAMPLE 8:__ *Modified Hellinger-Reissner functional*

In this example a *concentrated force* is acting on the midpoint ($x_1 = 0, x_2 = 0$) of the plate.

Type of elements: *quadratic elements*

Shape functions:

$M_{\alpha\beta}$, u_3, u_α *linear*

$N_{\alpha\beta}$ *constant*

Number of unknowns: 171

This functional differs from that of example 2 only by the load term. Here, the concentrated force must be normalized by

$$\bar{p}^* = \frac{Eh^4}{a^2} p^* .$$

The functional is

$$\begin{aligned} I_{RS} = &-\int_A W_c(N_{\kappa\beta}, M_{\kappa\beta})\,dA \quad - p^* u_3(0,0) \\ &+\int_A M_{\kappa\beta,\beta}\, u_{3,\kappa}\, dA \quad + \int_A \tfrac{1}{2} N_{\kappa\beta}\, u_{3,\kappa}\, u_{3,\beta}\, dA \\ &+\int_\Gamma \langle N_{\nu\nu}\rangle u_\nu\, d\sigma \quad + \int_\Gamma \langle N_{t\nu}\rangle u_t\, d\sigma . \end{aligned} \qquad (8.2.20)$$

The results are demonstrated in figure 29 to 34.

8.3. An example for the clamped plate

It shall be supposed that all edges of the supported plate are movable for in-plane directions. This case requires the following boundary conditions for Reissner's functional:

On C_1:
$$\left.\begin{aligned} u_3 &= 0 \\ \beta_\nu &= 0 \end{aligned}\right\} \qquad (8.3.1)$$

On C_2:
$$\left.\begin{aligned} N_{\alpha\beta} &= 0 \\ M_{\nu\nu} &= 0 \end{aligned}\right\} \qquad (8.3.2)$$

EXAMPLE 9: *Modified Hellinger-Reissner functional*

Type of elements: *quadratic elements*

Shape functions:

$M_{\alpha\beta}$, u_3, u_α *linear*

$N_{\alpha\beta}$ *constant*

Number of unknowns: 184

The difference between the present functional and functional I_{R2} exists in additional boundary integrals.

$$
\begin{aligned}
I_{R6} = & -\int_A W_c(N_{\alpha\beta}, M_{\alpha\beta})\, dA && -\int_A p^* u_3\, dA \\
& +\int_A M_{\alpha\beta,\beta}\, u_{3,\alpha}\, dA && +\int_A \tfrac{1}{2} N_{\alpha\beta}\, u_{3,\alpha} u_{3,\beta}\, dA \\
& +\int_{C_2} N_{\nu\nu} u_\nu\, ds && +\int_{C_2} N_{t\nu} u_t\, ds \\
& +\int_\Gamma \langle N_{\nu\nu}\rangle u_\nu\, d\sigma && +\int_\Gamma \langle N_{t\nu}\rangle u_t\, d\sigma
\end{aligned}
\qquad (8.3.3)
$$

A graphical representation of the results is shown in figure 35 to 40.

9. AN INCREMENTAL VARIATIONAL FORMULATION

In this method the actual load of the system is divided into a certain numer of increments. Then, after each increment a linear calculation follows by solving a system of linear algebraic equations. In doing so, each time the real solution of the nonlinear system will be reached subsequently by an iterative process involving all nonlinear terms [46], [50]. This incremental formulation in Lagrangean approach shall be demonstrated by means of functional I_{R2}. After applying Gauß' theorem this functional has been established from functional (4.1). Consequently, the variational statements can be derived immediately with reference to (4.1). According to the kind of shape functions of example 2 the following discontinuities exist:

On C_1:
$$\overset{*}{\beta}_\nu - \beta_\nu \neq 0 \tag{9.1}$$

On Γ :
$$\left.\begin{aligned} \langle \beta_\nu \rangle &\neq 0 \\ \langle N_{\nu\nu} \rangle &\neq 0 \\ \langle N_{t\nu} \rangle &\neq 0 \end{aligned}\right\} \tag{9.2}$$

Other discontinuities do not appear in (4.1). If integration by parts is performed again due to (7.1.6), one gets with (9.1) and (9.2) the functional:

$$\begin{aligned} I_{R2} = &-\int_A W_{cM}(N^{\alpha\beta})\,dA && -\int_A W_{cB}(M^{\alpha\beta})\,dA \\ &-\int_A M^{\alpha\beta} u_3|_{\alpha\beta}\,dA && +\int_A \tfrac{1}{2} N^{\alpha\beta} u_{3,\alpha} u_{3,\beta}\,dA \\ &-\int_A N^{\alpha\beta}|_\beta u_\alpha\,dA && -\int_A \overset{*}{p} u_3\,dA \\ &-\int_{C_2} \overset{*}{P}_{n\nu} u_3\,ds && -\int_{C_2} \overset{*}{K}_\nu \beta_\nu\,ds \\ &-\int_{C_2} (\overset{*}{P}_{\nu\nu} - N_{\nu\nu}) u_\nu\,ds && -\int_{C_2} (\overset{*}{P}_{t\nu} - N_{t\nu}) u_t\,ds \end{aligned}$$

$$+\int_{C_1} u_\nu^* N_{\nu\nu}\, ds \qquad + \int_{C_1} u_t^* N_{t\nu}\, ds$$

$$+\int_{C_1} (\beta_\nu^* - \beta_\nu) M_{\nu\nu}\, ds \qquad - \int_{\Gamma} \langle \beta_\nu \rangle M_{\nu\nu}\, d\sigma$$

$$+\int_{\Gamma} \langle N_{\nu\nu} \rangle u_\nu\, d\sigma \qquad + \int_{\Gamma} \langle N_{t\nu} \rangle u_t\, d\sigma$$

$$- [\, K_t^* u_3 \,]_{C_2} \tag{9.3}$$

The special case of the *simply supported plate* appears by deleting appropriate terms. For the sake of completeness this shall be not done yet for the present.

It is assumed now that the state of displacements and that of stresses are known under the present load. In this state the loads are increased by an increment $\Delta(\ldots)$. In consequence, the field quantities increase incrementally too and also the value of the functional.

Field quantities, which are known already in the reference state are distinguished by an index o, whereas increments are not indicated separately.

$$I_{R2} + \Delta I_{R2}$$

$$= -\int_A W_{CM}(\overset{\circ}{N}{}^{\alpha\beta} + N^{\alpha\beta})\, dA \quad - \int_A W_{CB}(\overset{\circ}{M}{}^{\alpha\beta} + M^{\alpha\beta})\, dA$$

$$-\int_A (\overset{\circ}{M}{}^{\alpha\beta} + M^{\alpha\beta})(\overset{\circ}{u}_3 + u_3)|_{\alpha\beta}\, dA$$

$$+\int_A \tfrac{1}{2}(\overset{\circ}{N}{}^{\alpha\beta} + N^{\alpha\beta})(\overset{\circ}{u}_3 + u_3)_{,\alpha}(u_3 + u_3)_{,\beta}\, dA$$

$$-\int_A (\overset{\circ}{N}{}^{\alpha\beta} + N^{\alpha\beta})|_\beta (\overset{\circ}{u}_\alpha + u_\alpha)\, dA$$

$$-\int_A (p^* + \Delta p^*)(\overset{\circ}{u}_3 + u_3)\, dA$$

$$-\int_{C_2} (P_{n\nu}^* + \Delta P_{n\nu})(\overset{\circ}{u}_3 + u_3)\, ds$$

$$-\int_{C_2} (K_\nu^* + \Delta K_\nu^*)(\overset{\circ}{\beta}_\nu + \beta_\nu)\, ds$$

$$+\int_{C_1} (u_\nu^* + \Delta u_\nu^*)(\overset{\circ}{N}_{\nu\nu} + N_{\nu\nu})\,ds$$

$$+\int_{C_1} (u_t^* + \Delta u_t^*)(\overset{\circ}{N}_{t\nu} + N_{t\nu})\,ds$$

$$-\int_{C_1} [(\beta_\nu^* + \Delta\beta_\nu^*) - (\overset{\circ}{\beta}_\nu + \beta_\nu)](\overset{\circ}{M}_{\nu\nu} + M_{\nu\nu})\,ds$$

$$-\int_{C_2} [(P_{\nu\nu}^* + \Delta P_{\nu\nu}^*) - (\overset{\circ}{N}_{\nu\nu} + N_{\nu\nu})](\overset{\circ}{u}_\nu + u_\nu)\,ds$$

$$-\int_{C_2} [(P_{t\nu}^* + \Delta P_{t\nu}^*) - (\overset{\circ}{N}_{t\nu} + N_{t\nu})](\overset{\circ}{u}_t + u_t)\,ds$$

$$-\int_{\Gamma} \langle \overset{\circ}{\beta}_\nu + \beta_\nu \rangle (\overset{\circ}{M}_{\nu\nu} + M_{\nu\nu})\,d\sigma$$

$$+\int_{\Gamma} \langle \overset{\circ}{N}_{\nu\nu} + N_{\nu\nu} \rangle (\overset{\circ}{u}_\nu + u_\nu)\,d\sigma$$

$$+\int_{\Gamma} \langle \overset{\circ}{N}_{t\nu} + N_{t\nu} \rangle (\overset{\circ}{u}_t + u_t)\,d\sigma$$

$$-[(K_t^* + \Delta K_t^*)(\overset{\circ}{u}_3 + u_3)]_{C_2} \qquad (9.4)$$

Obviously, there is no doubt that the functions of energy density are continuously differentiable with respect to their arguments, so that a taylor-expansion up to the second order is possible. Without that, a further expansion is not performable, since the energy functions are quadratic forms.

$$W_{CM}(\overset{\circ}{N}{}^{\alpha\beta} + N^{\alpha\beta}) = W_{CM}(\overset{\circ}{N}{}^{\alpha\beta}) + \frac{\partial W_{CM}}{\partial \overset{\circ}{N}{}^{\alpha\beta}} N^{\alpha\beta} + \frac{1}{2}\frac{\partial^2 W_{CM}}{\partial \overset{\circ}{N}{}^{\alpha\beta}\partial \overset{\circ}{N}{}^{\varrho\lambda}} N^{\alpha\beta} N^{\varrho\lambda} \qquad (9.5)$$

$$W_{CB}(\overset{\circ}{M}{}^{\alpha\beta} + M^{\alpha\beta}) = W_{CB}(\overset{\circ}{M}{}^{\alpha\beta}) + \frac{\partial W_{CB}}{\partial \overset{\circ}{M}{}^{\alpha\beta}} M^{\alpha\beta} + \frac{1}{2}\frac{\partial^2 W_{CB}}{\partial \overset{\circ}{M}{}^{\alpha\beta}\partial \overset{\circ}{M}{}^{\varrho\lambda}} M^{\alpha\beta} M^{\varrho\lambda} \qquad (9.6)$$

In any adjacent state the functional (9.3) attains a stationary value, if the first variation of the incremental functional (9.4) vanishes. After integration by parts and using relations (9.5) and (9.6), the first variation leads to:

$$\delta(I_{R2} + \Delta I_{R2})$$

$$= \Big\{ \int_A \Big[\frac{\partial W_{CM}}{\partial \mathring{N}^{\alpha\beta}} - \tfrac{1}{2}(\mathring{u}_\alpha|_\beta + \mathring{u}_\beta|_\alpha + \mathring{u}_{3,\alpha}\mathring{u}_{3,\beta}) \Big] \delta N^{\alpha\beta} dA$$

$$- \int_A \Big[\frac{\partial W_{CB}}{\partial \mathring{M}^{\alpha\beta}} + u_3|_{\alpha\beta} \Big] \delta M^{\alpha\beta} dA$$

$$- \int_A \Big[\mathring{M}^{\alpha\beta}|_{\alpha\beta} + (\mathring{N}^{\alpha\beta}\mathring{u}_{3,\alpha})_{,\alpha} + p^* \Big] \delta u_3 \, dA$$

$$- \int_A \mathring{N}^{\alpha\beta}|_\beta \, \delta u_\alpha \, dA$$

$$+ \int_{C_1} (u_\nu^* - \mathring{u}_\nu) \delta N_{\nu\nu} ds \quad + \int_{C_1} (u_t^* - u_t) \delta N_{t\nu} ds$$

$$- \int_{C_1} (P_{n\nu}^* - \mathring{P}_{n\nu}) \delta u_3 ds \quad - \int_{C_2} (K_\nu^* - \mathring{K}_\nu) \delta\beta_\nu ds$$

$$- \Big[(K_t^* - \mathring{M}_{t\nu}) \delta u_3 \Big]_{C_2}$$

$$- \int_{C_1} (\beta_\nu^* - \beta_\nu) \delta M_{\nu\nu} ds \quad - \int_\Gamma \langle \mathring{\beta}_\nu \rangle \delta M_{\nu\nu} ds$$

$$- \int_{C_2} (P_{\nu\nu}^* - N_{\nu\nu}) \delta u_\nu ds \quad - \int_{C_2} (P_{t\nu}^* - N_{t\nu}) \delta u_t ds$$

$$+ \int_\Gamma \langle \mathring{N}_{\nu\nu} \rangle \delta u_\nu d\sigma \quad + \int_\Gamma \langle \mathring{N}_{t\nu} \rangle \delta u_t d\sigma \Big\} +$$

$$+ \Big\{ - \int_A \Big[\frac{\partial^2 W_{CM}}{\partial \mathring{N}^{\alpha\beta} \partial \mathring{N}^{\varphi\lambda}} N^{\varphi\lambda} - \tfrac{1}{2}(u_\alpha|_\beta + u_\beta|_\alpha + u_{3,\alpha}u_{3,\beta} - \mathring{u}_{3,\alpha}u_{3,\beta}) \Big] \delta N^{\alpha\beta} dA$$

$$- \int_A \Big[\frac{\partial^2 W_{CB}}{\partial \mathring{M}^{\alpha\beta} \partial \mathring{M}^{\varphi\lambda}} M^{\varphi\lambda} + \mathring{u}_3|_{\alpha\beta} \Big] \delta M^{\alpha\beta} dA$$

$$- \int_A N^{\alpha\beta}|_\beta \, \delta u_\alpha \, dA$$

$$- \int_A \Big\{ M^{\alpha\beta}|_{\alpha\beta} + \big[\mathring{N}^{\alpha\beta} u_{3,\beta} + N^{\alpha\beta}(\mathring{u}_{3,\beta} + u_{3,\beta}) \big]\big|_\alpha + \Delta p^* \Big\} \delta u_3 \, dA$$

$$
\left.
\begin{aligned}
&+\int_{C_1} (\Delta u_\nu^* - u_\nu)\, \delta N_{\nu\nu}\, ds \qquad +\int_{C_1} (\Delta u_t^* - u_t)\, \delta N_{t\nu}\, ds \\
&-\int_{C_1} \{ \Delta p_{m\nu}^* - \mathring{p}_{m\nu} - [\mathring{N}^{\alpha\beta} u_{3,\beta} + N^{\alpha\beta}(\mathring{u}_{3,\beta} + u_{3,\beta})]\, \nu_\alpha \}\, \delta u_3\, dA \\
&-\int_{C_2} (\Delta M_{\nu\nu}^* - M_{\nu\nu})\, \delta\beta_\nu\, ds - [(\Delta K_t^* - M_{t\nu})\, \delta u_3]_{C_2} \\
&+\int_{\Gamma} \langle p_{m\nu} + [\mathring{N}^{\alpha\beta} u_{3,\beta} + N^{\alpha\beta}(\mathring{u}_{3,\beta} + u_{3,\beta})]\, \nu_\alpha \rangle\, \delta u_3\, d\sigma \\
&-\int_{C_1} (\Delta\beta_\nu^* - \beta_\nu)\, \delta M_{\nu\nu}\, ds \qquad +\int_{\Gamma} \langle \beta_\nu \rangle\, \delta M_{\nu\nu}\, d\sigma \\
&-\int_{C_2} (\Delta p_{\nu\nu}^* - N_{\nu\nu})\, \delta u_\nu\, ds \qquad -\int_{C_2} (\Delta p_{t\nu}^* - N_{t\nu})\, \delta u_t\, ds \\
&+\int_{\Gamma} \langle N_{\nu\nu} \rangle\, \delta u_\nu\, d\sigma \qquad +\int_{\Gamma} \langle N_{t\nu} \rangle\, \delta u_t\, d\sigma
\end{aligned}
\right\} \quad (9.7)
$$

As you see, the first variation yields the *Euler-Lagrange-equations*, all *boundary* and *interelement boundary conditions* of the original and of the incrementally deformed configuration. If boundary conditions of the simply supported plate are taken into considereation, the functional (8.2.2) can be formulated directly in incremental form. This formulation is then more simplified, if constant terms are dropped, which do not contribute anything to the variation. Thus, the following functional can be put as a basis for numerical calculations:

$$
\begin{aligned}
&I_{R2} + \Delta I_{R2} \\
&= -\int_A \Big[\frac{\partial W_{CM}}{\partial \mathring{N}^{\alpha\beta}} N^{\alpha\beta} + \frac{1}{2} \frac{\partial^2 W_{CM}}{\partial \mathring{N}^{\alpha\beta} \partial \mathring{N}^{\varrho\lambda}} N^{\alpha\beta} N^{\varrho\lambda} \Big]\, dA \\
&\quad -\int_A \Big[\frac{\partial W_{CB}}{\partial \mathring{M}^{\alpha\beta}} M^{\alpha\beta} + \frac{1}{2} \frac{\partial^2 W_{CB}}{\partial \mathring{M}^{\alpha\beta} \partial \mathring{M}^{\varrho\lambda}} M^{\alpha\beta} M^{\varrho\lambda} \Big]\, dA \\
&\quad -\int_A (p^* + \Delta p^*)\, \mathring{u}_3\, dA \\
&\quad +\int_A (\mathring{M}^{\alpha\beta}|_\beta\, u_{3,\alpha} + M^{\alpha\beta}|_\beta\, \mathring{u}_{3,\alpha} + M^{\alpha\beta}|_\beta\, u_{3,\alpha}]\, dA
\end{aligned}
$$

$$+\int_A \left(\mathring{N}^{\alpha\beta} u_{3,\alpha} u_{3,\beta} + \tfrac{1}{2} \mathring{N}^{\alpha\beta} u_{3,\alpha} u_{3,\beta} + \tfrac{1}{2} N^{\alpha\beta} \mathring{u}_{3,\alpha} \mathring{u}_{3,\beta} + N^{\alpha\beta} \mathring{u}_{3,\alpha} u_{3,\beta} + \tfrac{1}{2} N^{\alpha\beta} u_{3,\alpha} u_{3,\beta} \right) dA$$

$$+\int_\Gamma \left[\langle \mathring{N}_{\nu\nu} \rangle u_\nu + \langle N_{\nu\nu} \rangle \mathring{u}_\nu + \langle N_{\nu\nu} \rangle u_\nu \right] d\sigma$$

$$+\int_\Gamma \left[\langle N_{t\nu} \rangle u_t + \langle N_{t\nu} \rangle \mathring{u}_t + \langle N_{t\nu} \rangle u_t \right] d\sigma \qquad (9.8)$$

Numerical calculations were performed for

$$\frac{p^*}{Eh^4/a^4} = 1$$

with five load increments, where two kinds of iteration have been applied within each increment. The first possibility requires the right side of the system of equations to be constant, in order to establish the iteration process by involving all nonlinear terms into the system matrix. The second possibilitiy consists in calculating with unchanged system matrix while iterating by the right side of the system of equations. Both iteration processes were stopped, when the maximum norm

$$\max \left| 1 - \frac{z_{i,j}}{z_{i,j-1}} \right| < 10^{-5} \qquad (9.9)$$

was fullfilled, wherein $z_{i,j}$ stands for the i-th component of the solution vector ascertained incrementally after the j-th iteration.
Figure 41 shows the graph of the midpoint deflection of the plate with iteration and without it as well.

10. REPRESENTATION OF NUMERICAL RESULTS

EXAMPLE 1: $\frac{p^*}{Eh^4/a^4} = 1$, $\nu = 0{,}3$

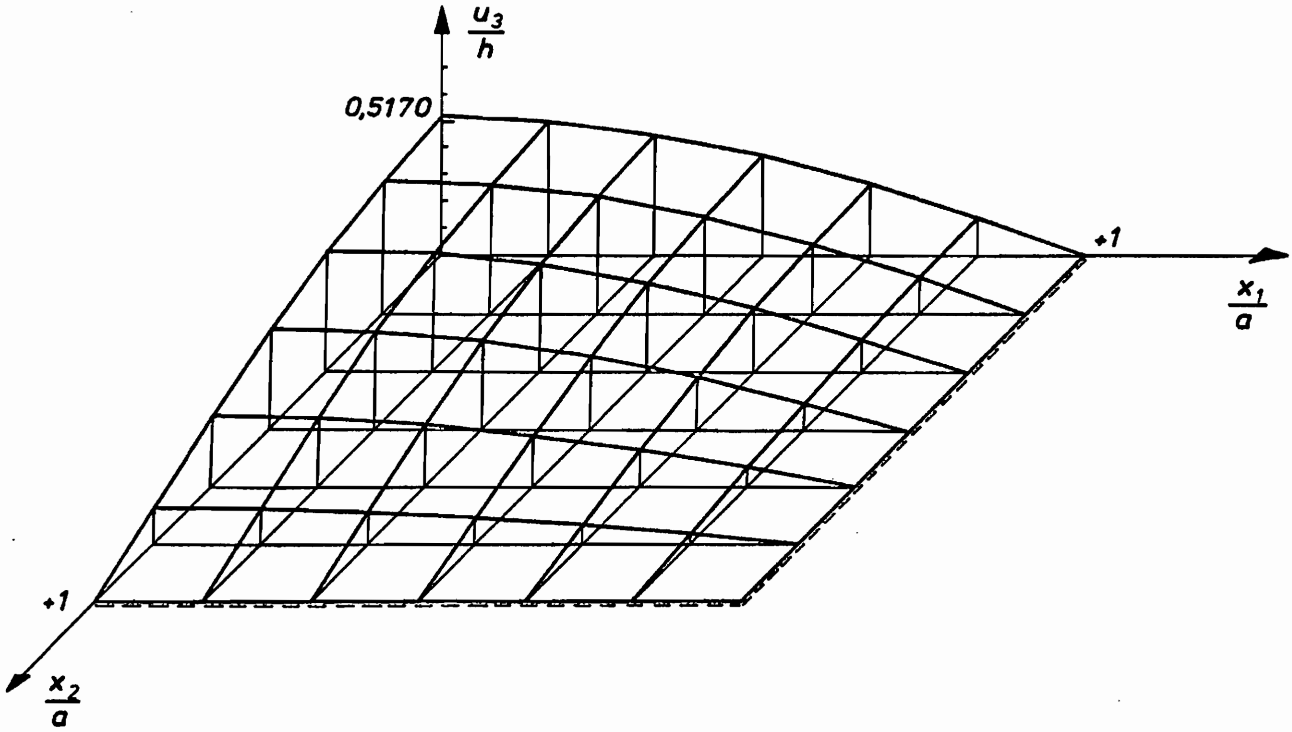

Fig. 6: Deflection

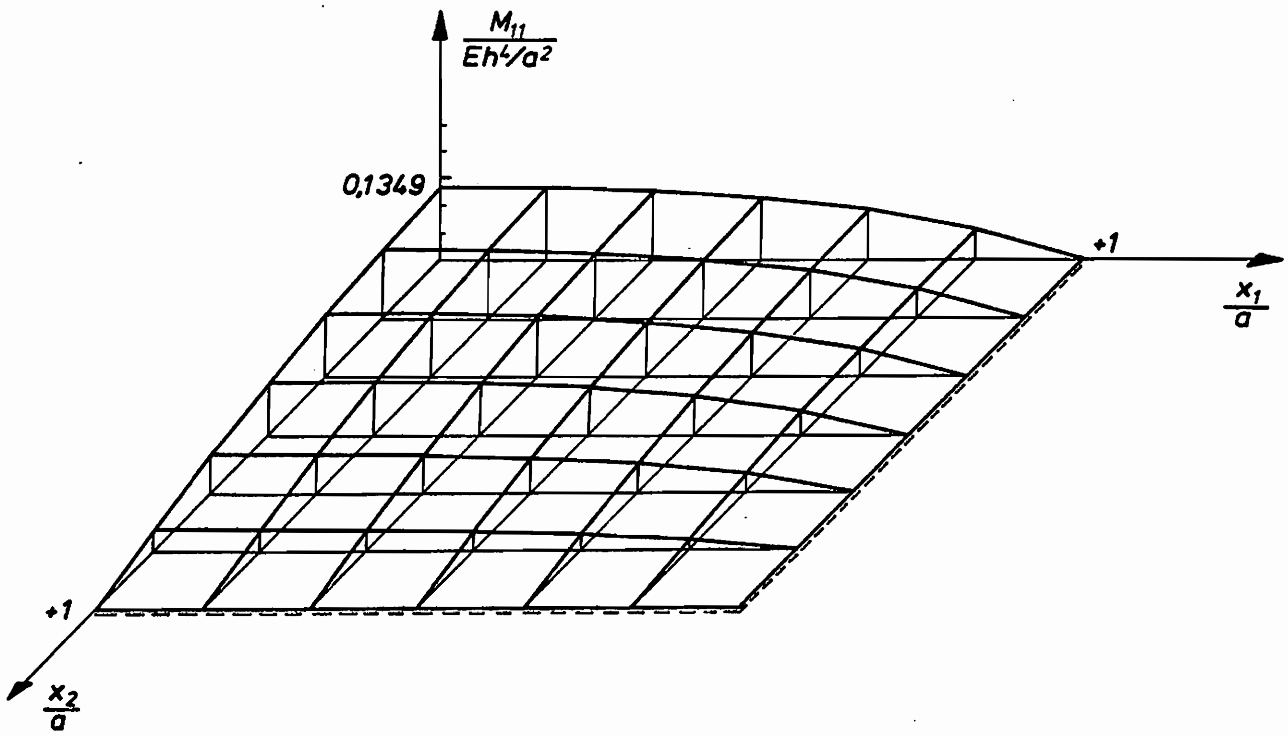

Fig. 7: Bending moment in x_1-direction

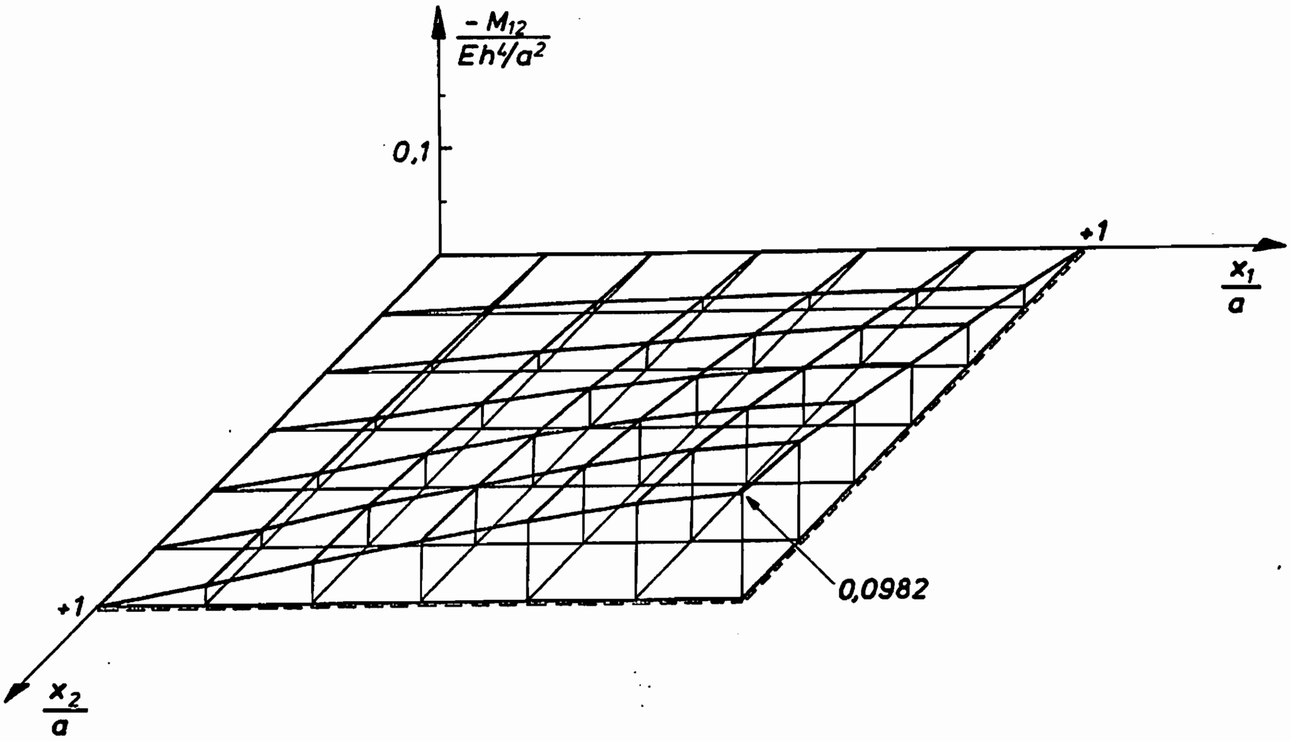

Fig. 8: Twisting moment

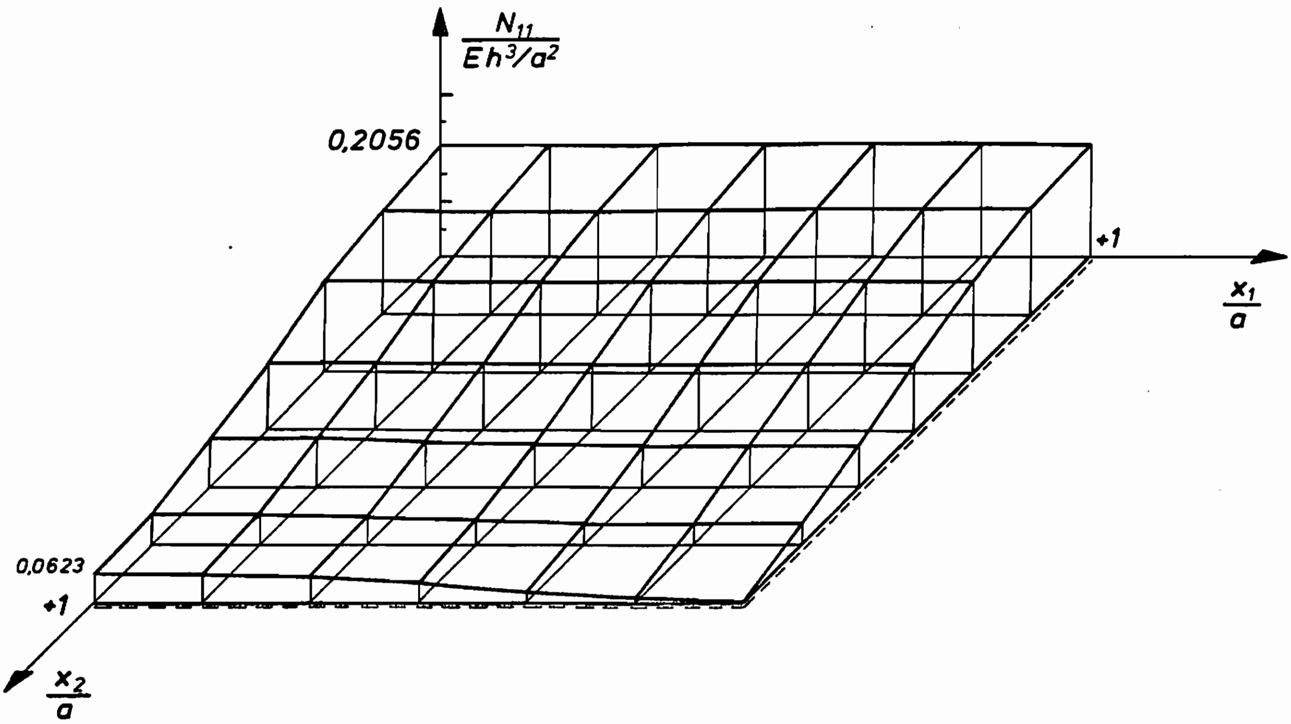

Fig. 9: Normal stresses in x_1-direction

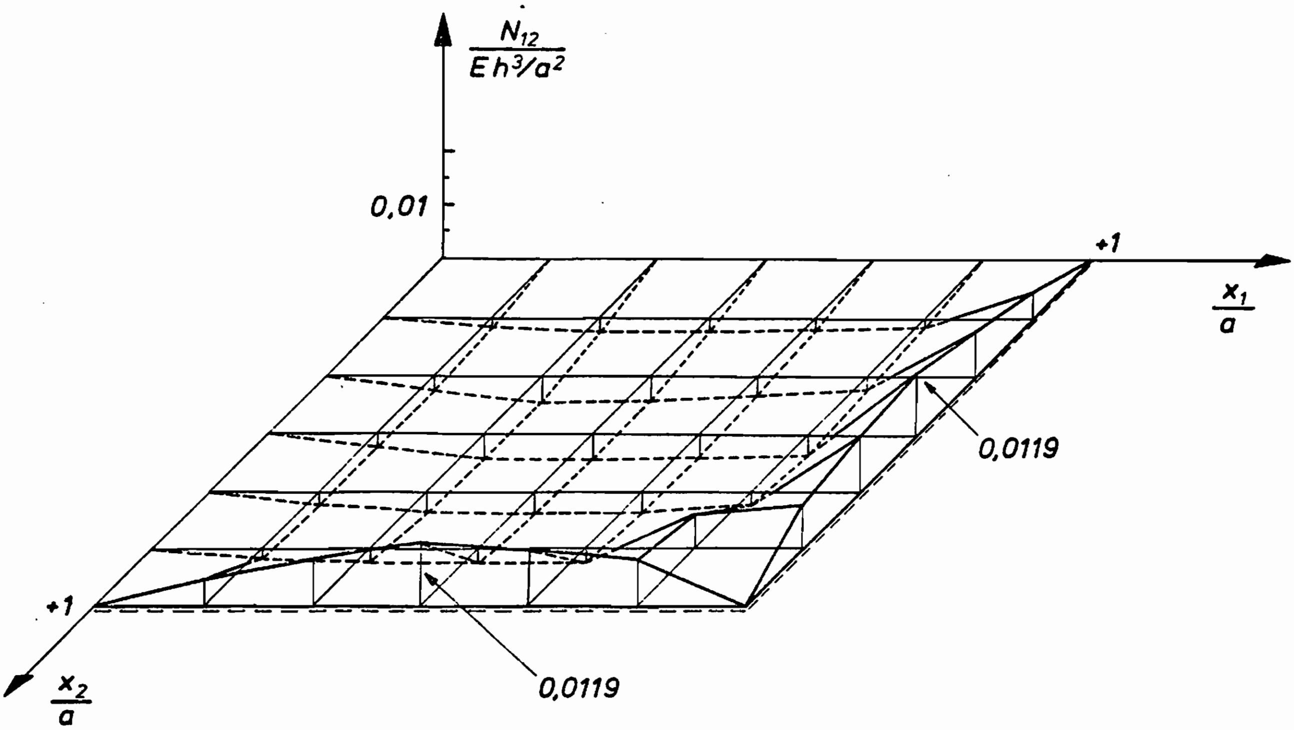

Fig. 10; Shear stresses

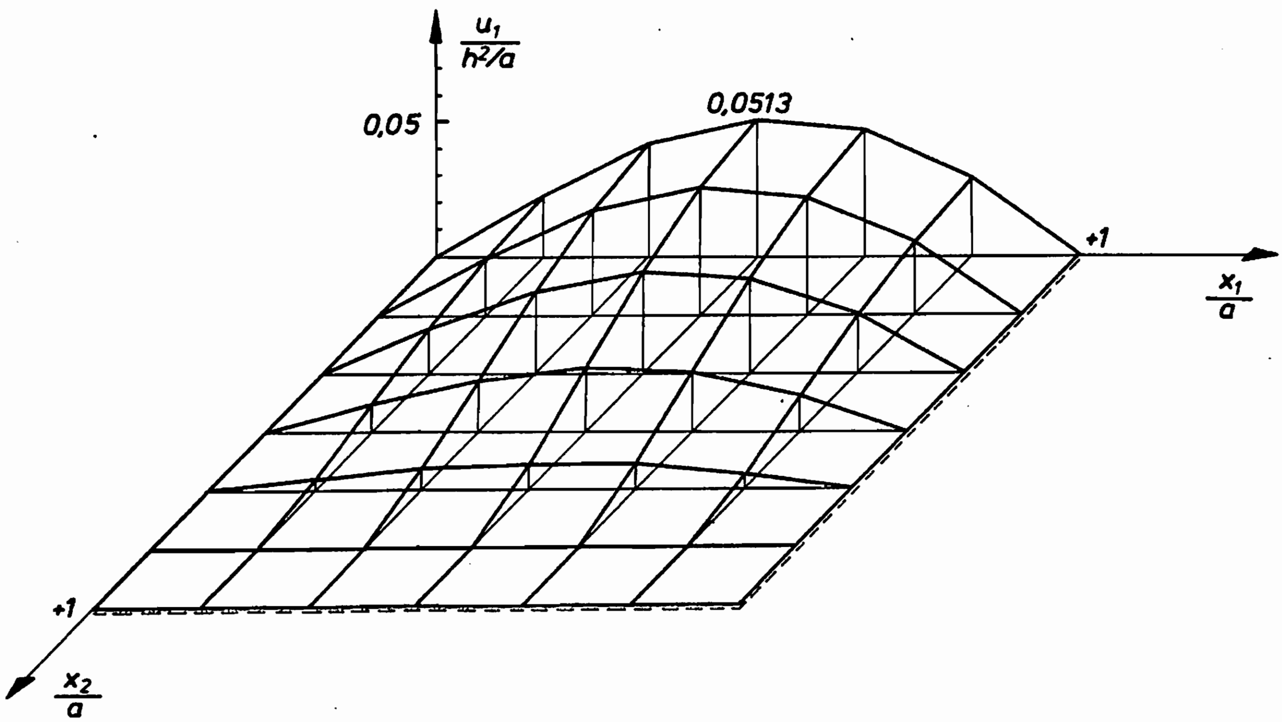

Fig. 11: In-plane displacements in x_1-direction

<u>*EXAMPLE 2:*</u> $\frac{p^*}{Eh^4/a^4} = 1$, $\nu = 0,3$

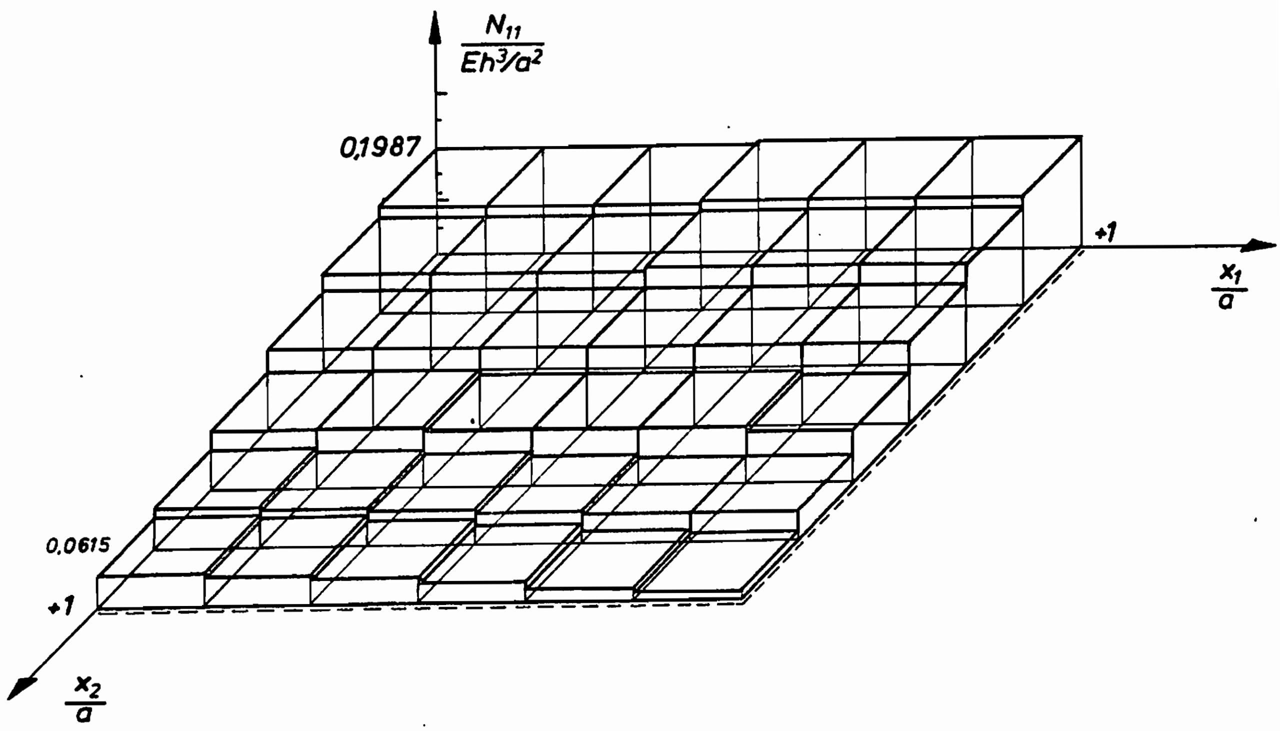

Fig. 12: Normal stresses in x_1-direction

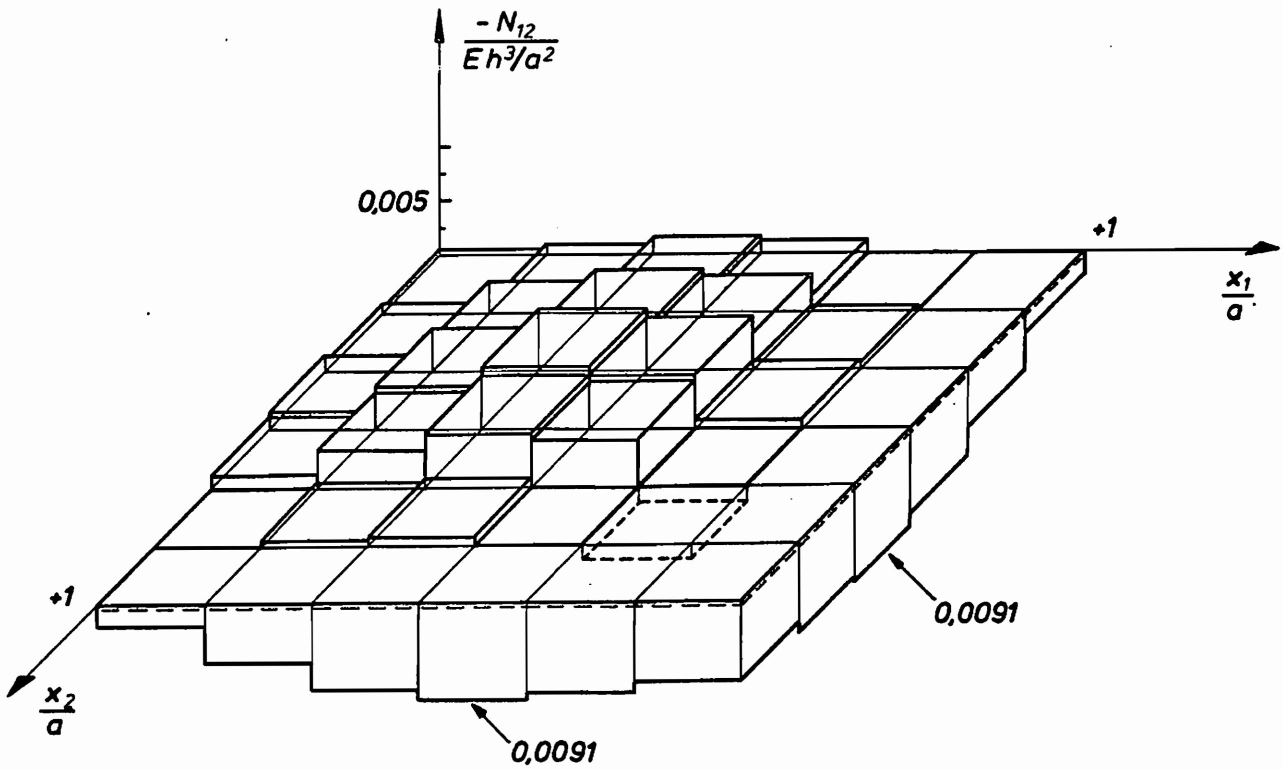

Fig. 13: Shear stresses

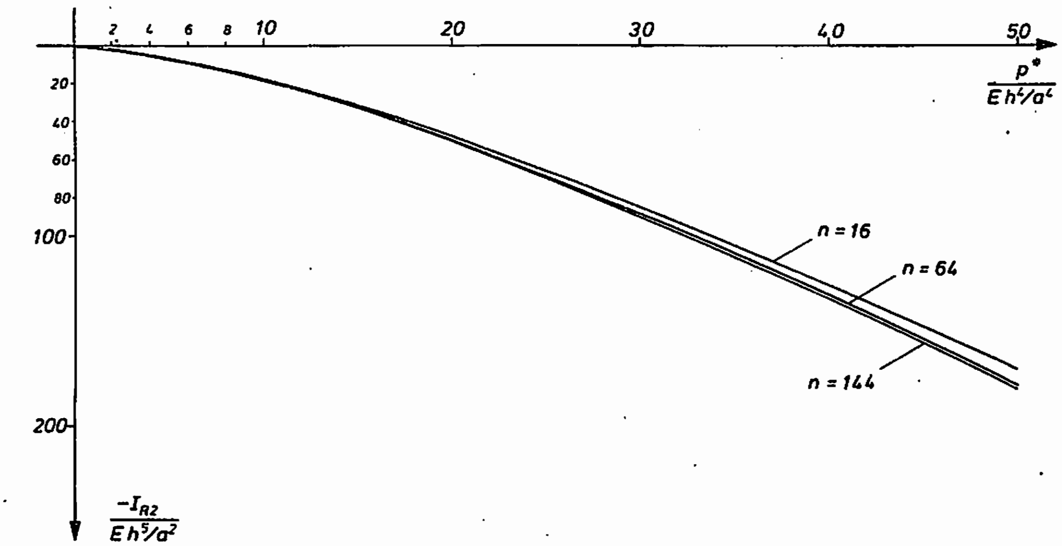

Fig. 14: Value of the functional in dependence of the load

EXAMPLE 3: $\frac{p^*}{Eh^4/a^4} = 1 \quad , \quad \nu = 0{,}3$

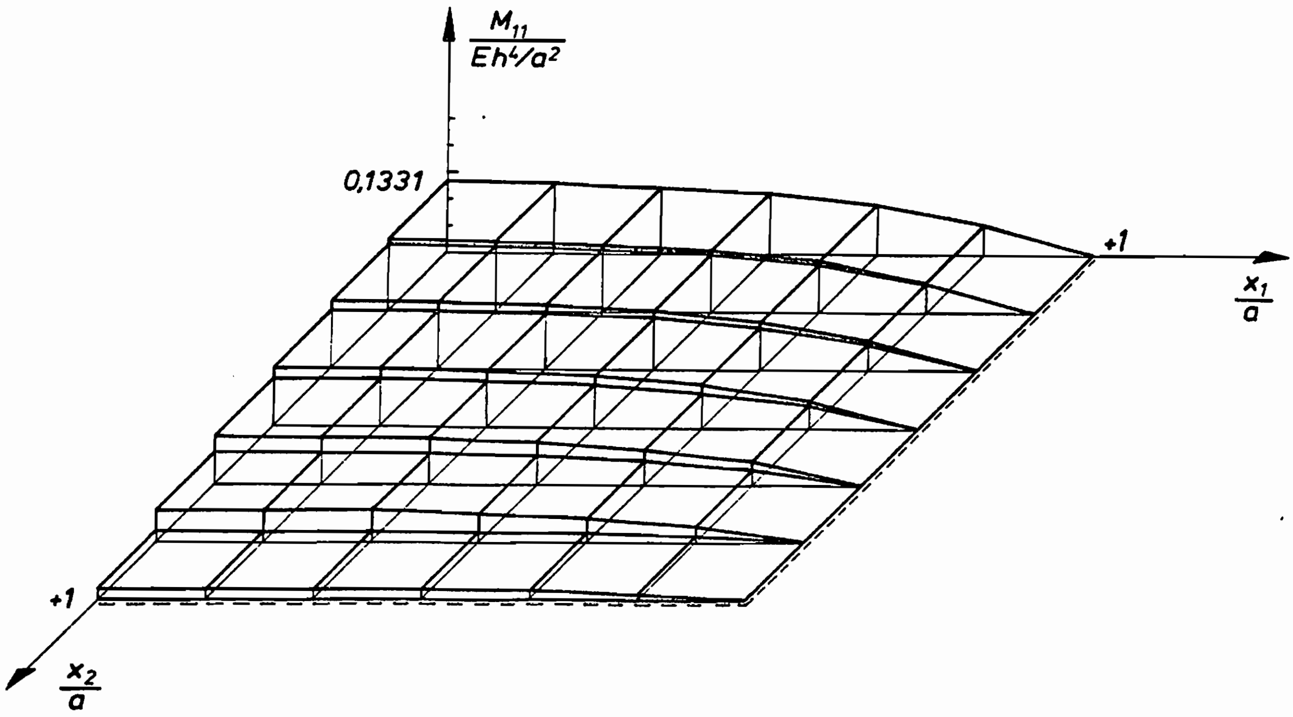

Fig. 15; Bending moment in x_1-direction

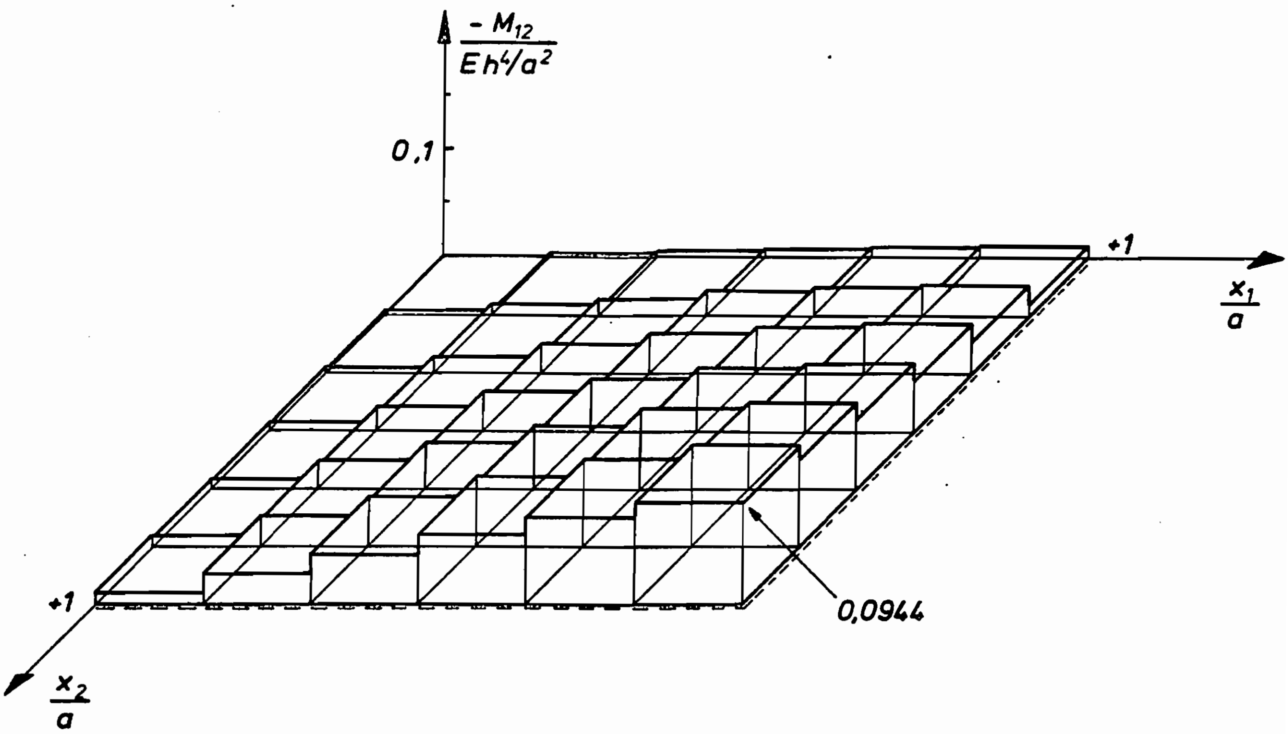

Fig. 16: Twisting moment

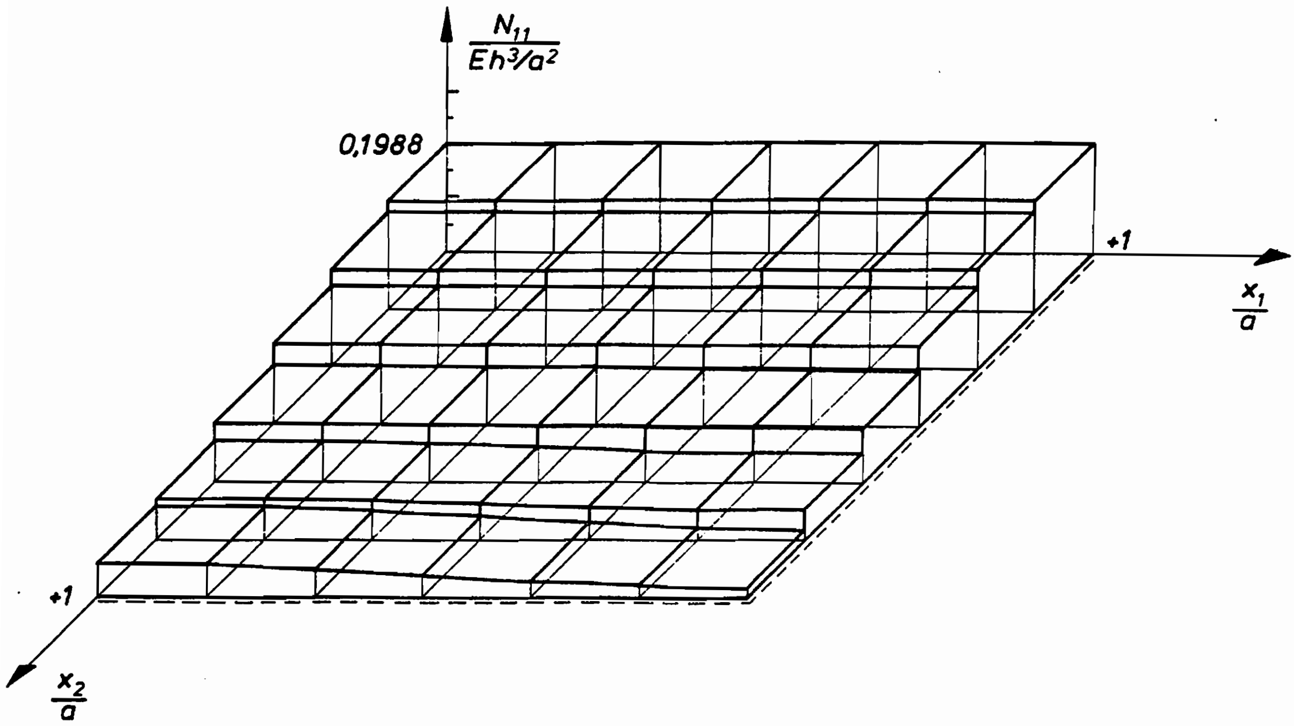

Fig. 17: Normal stresss in x_1-direction

EXAMPLE 4: $\frac{p^*}{Eh^4/a^4} = 1$, $= 0{,}3$

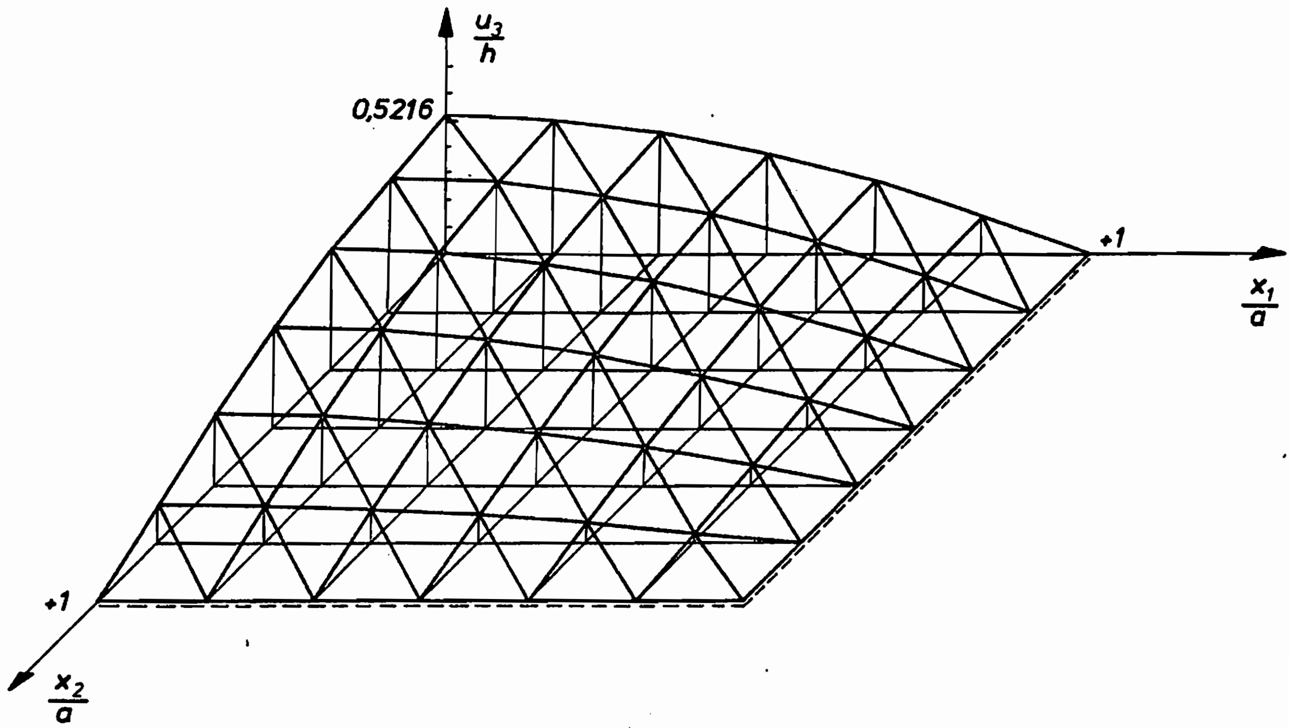

Fig. 18: Deflection

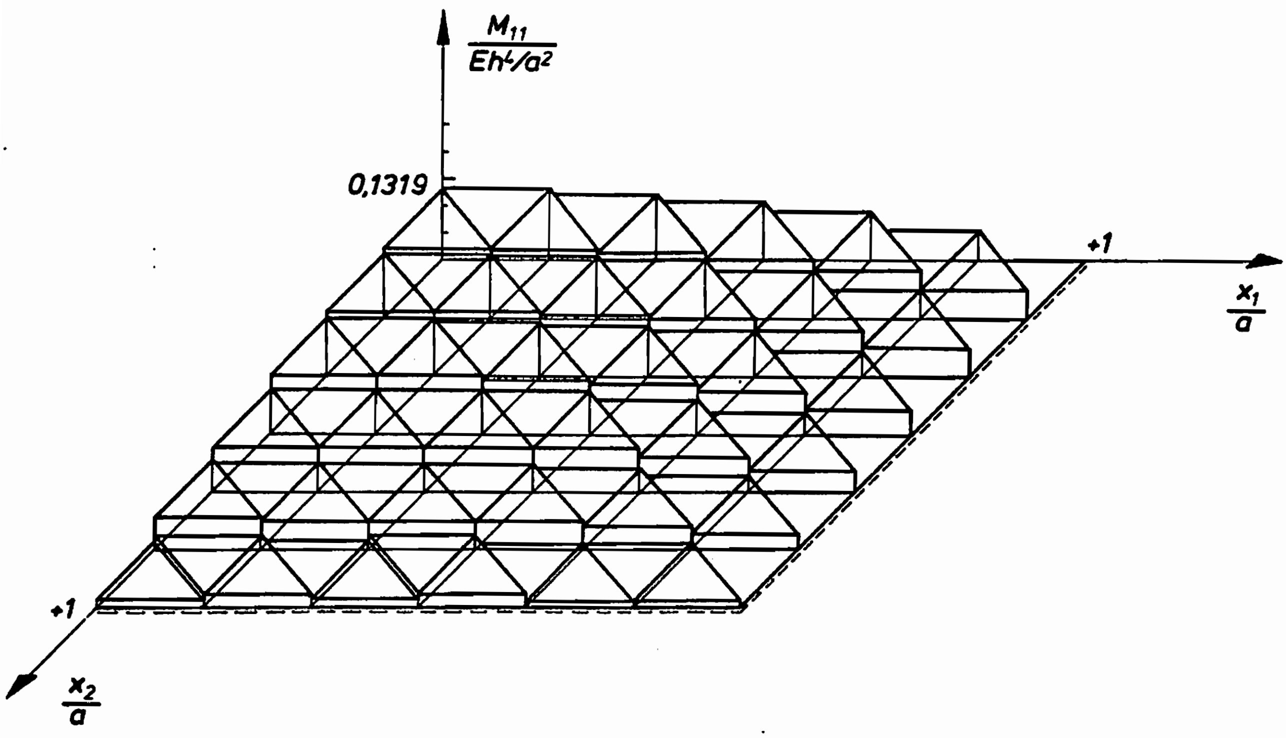

Fig. 19: Bending moment in x_1-direction

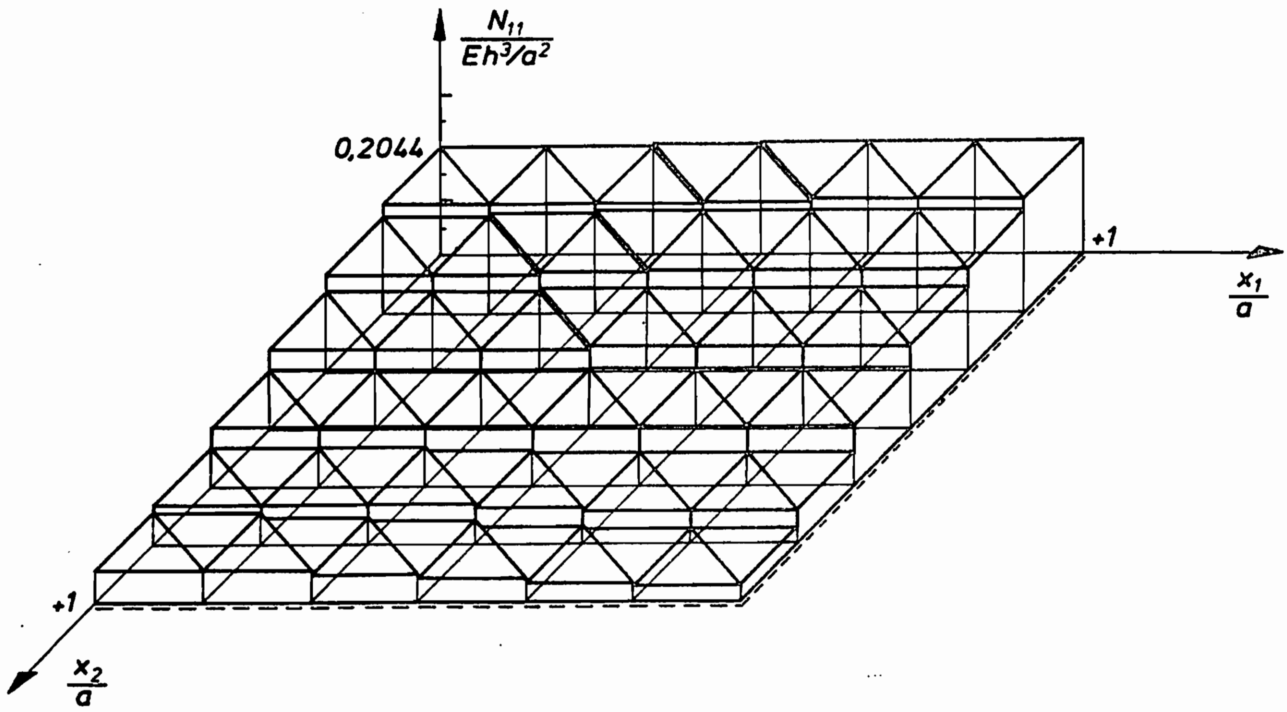

Fig. 20: Normal stresses in x_1-direction

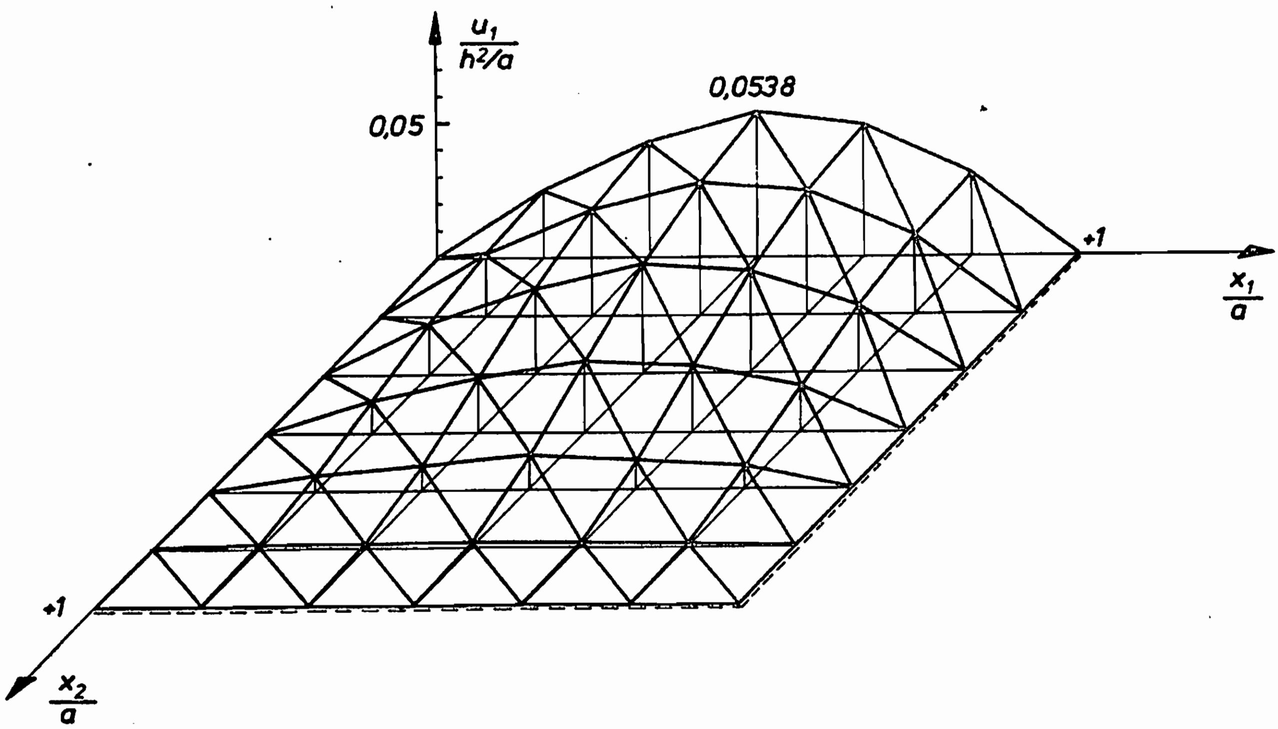

Fig. 21: In-plane displacements in x_1-direction

EXAMPLE 5: $\frac{p^*}{Eh^4/a^4} = 1 \quad , \quad \nu = 0{,}3$

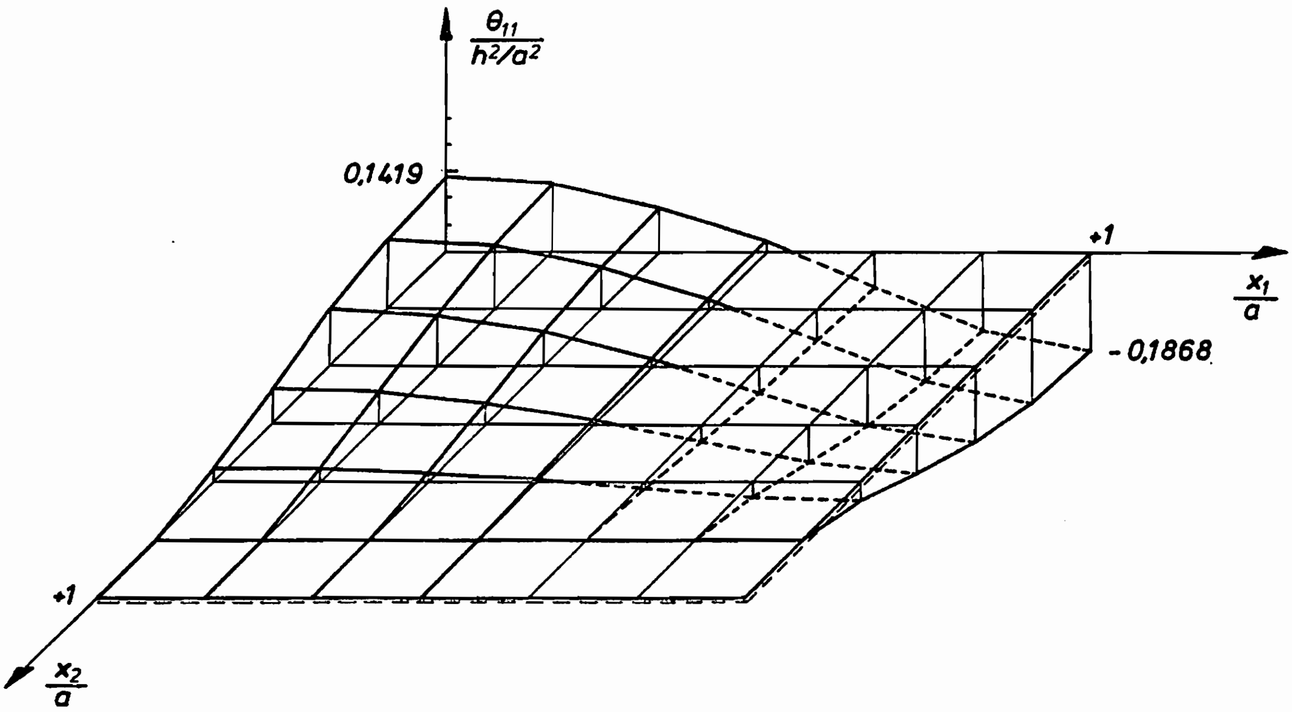

Fig. 22: Linear strains in x_1-direction

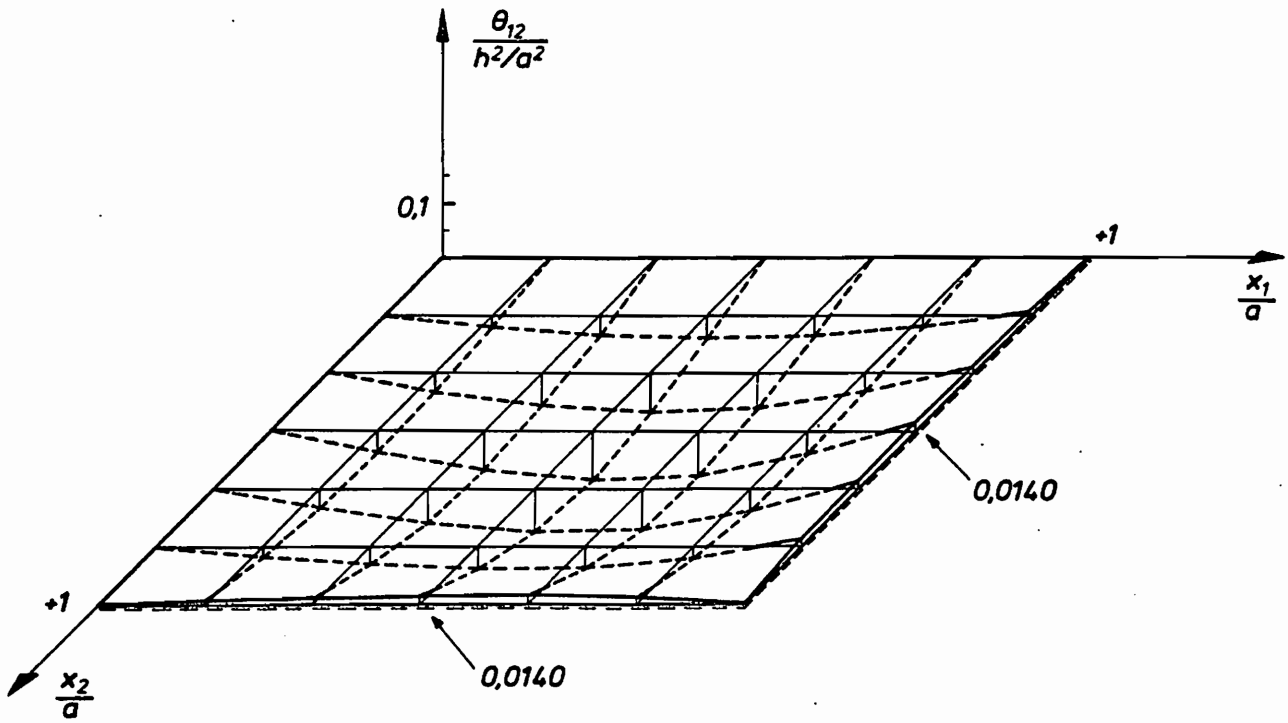

Fig. 23: Linear shear strains

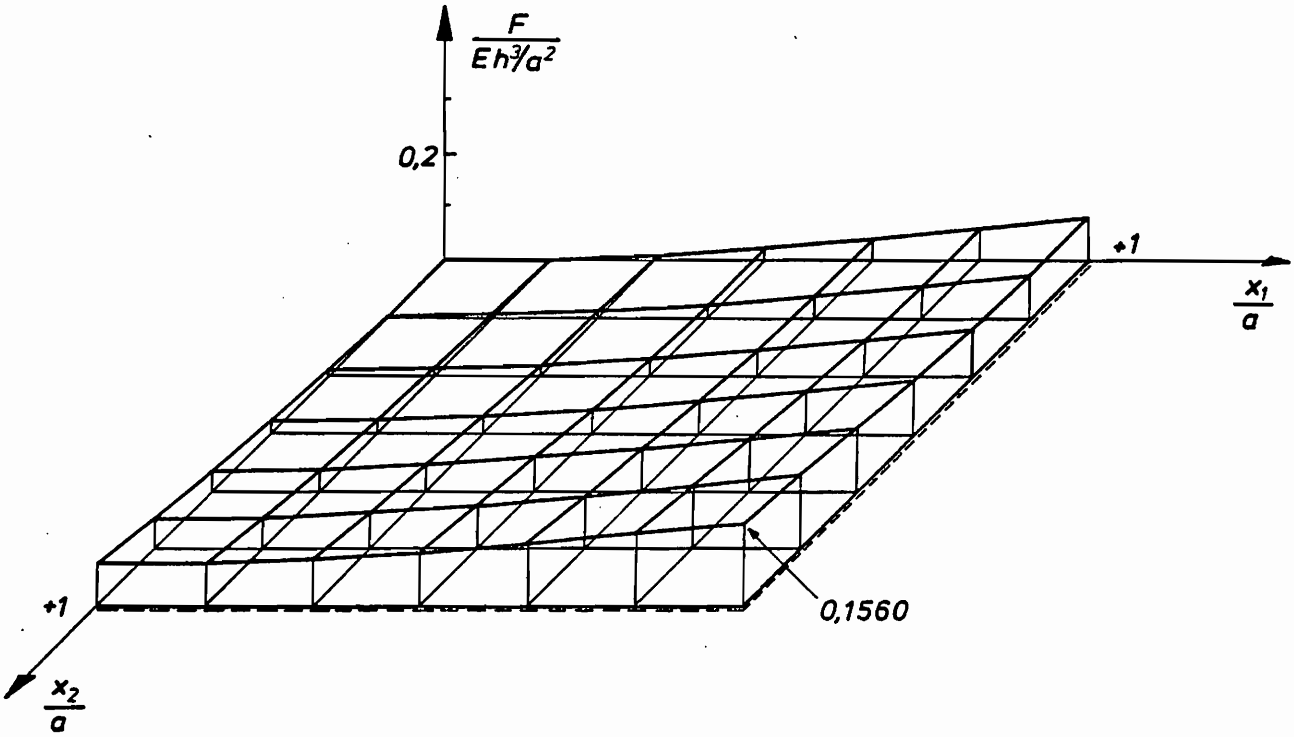

Fig. 24: Airy's stress function

EXAMPLE 6: $\frac{p^*}{Eh^4/a^4} = 1 \quad , \quad \nu = 0{,}3$

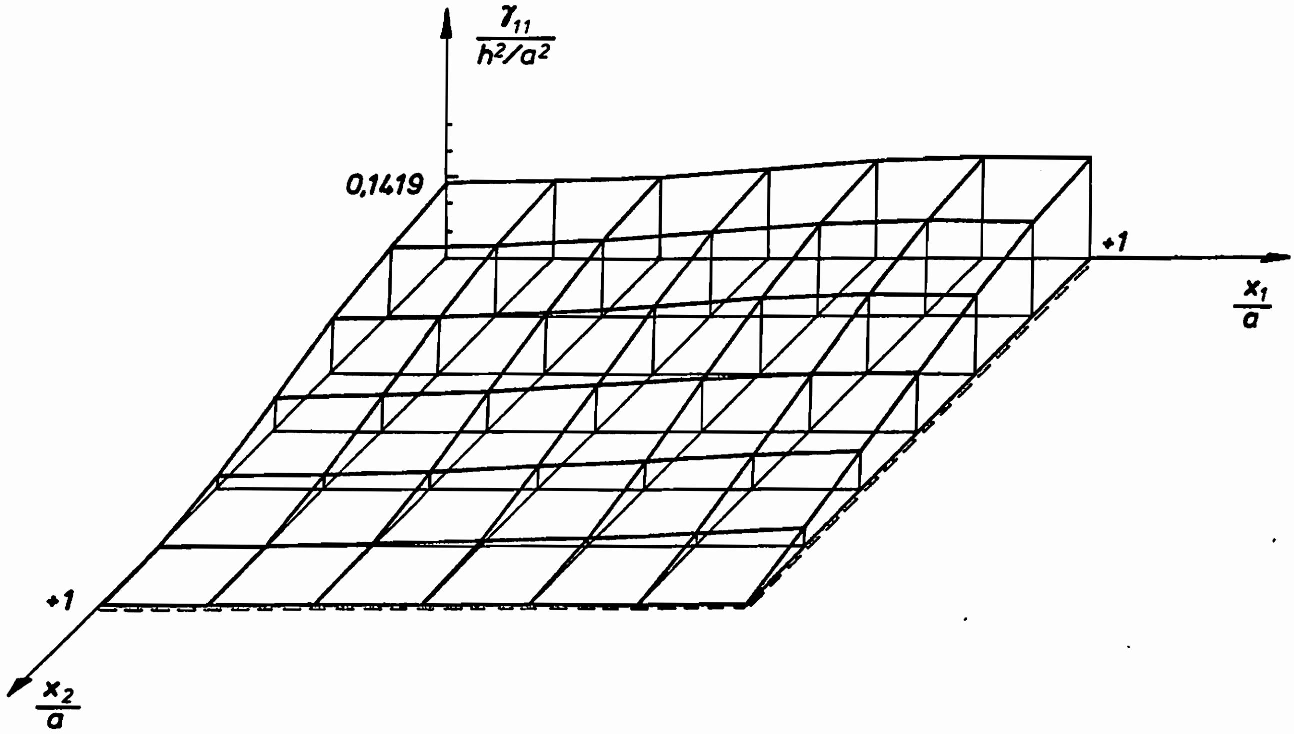

Fig. 25: Nonlinear strains in x_1-direction

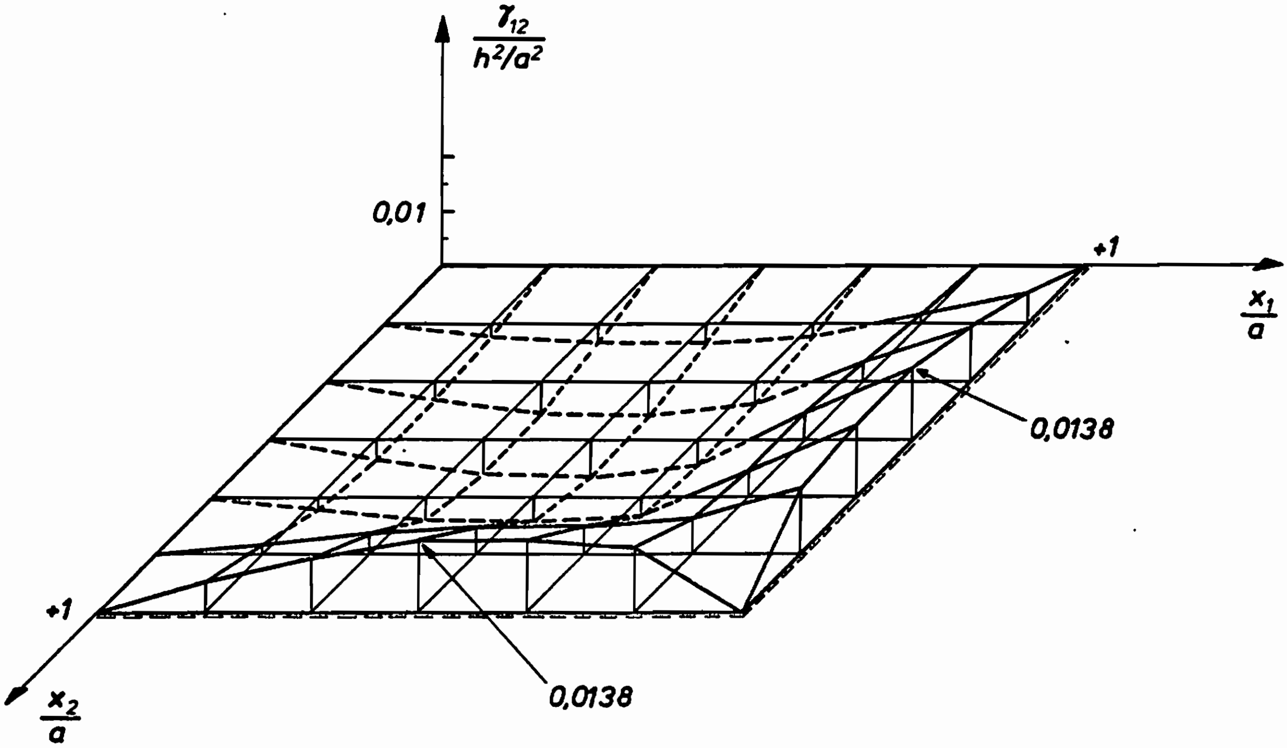

Fig. 26: Nonlinear shear strains

EXAMPLE 7: $\frac{p^*}{Eh^4/a^4} = 1$, $\nu = 0,3$

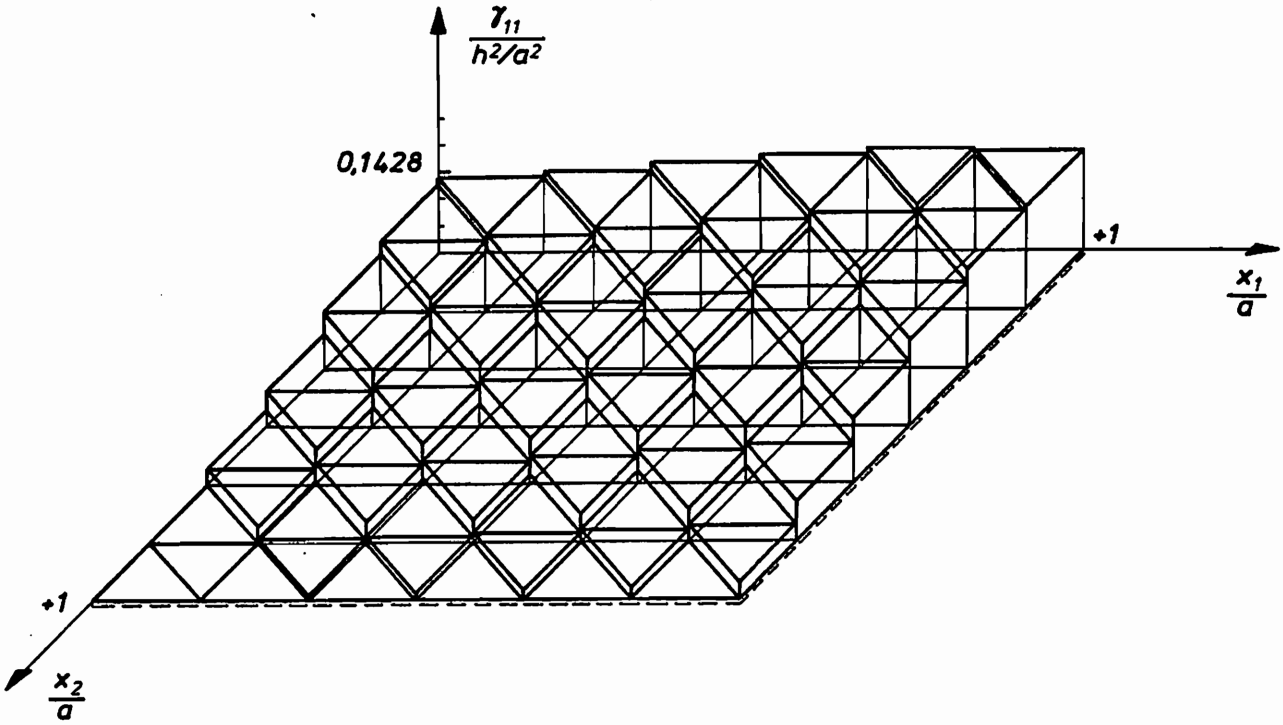

Fig. 27: Nonlinear strains in x_1-direction

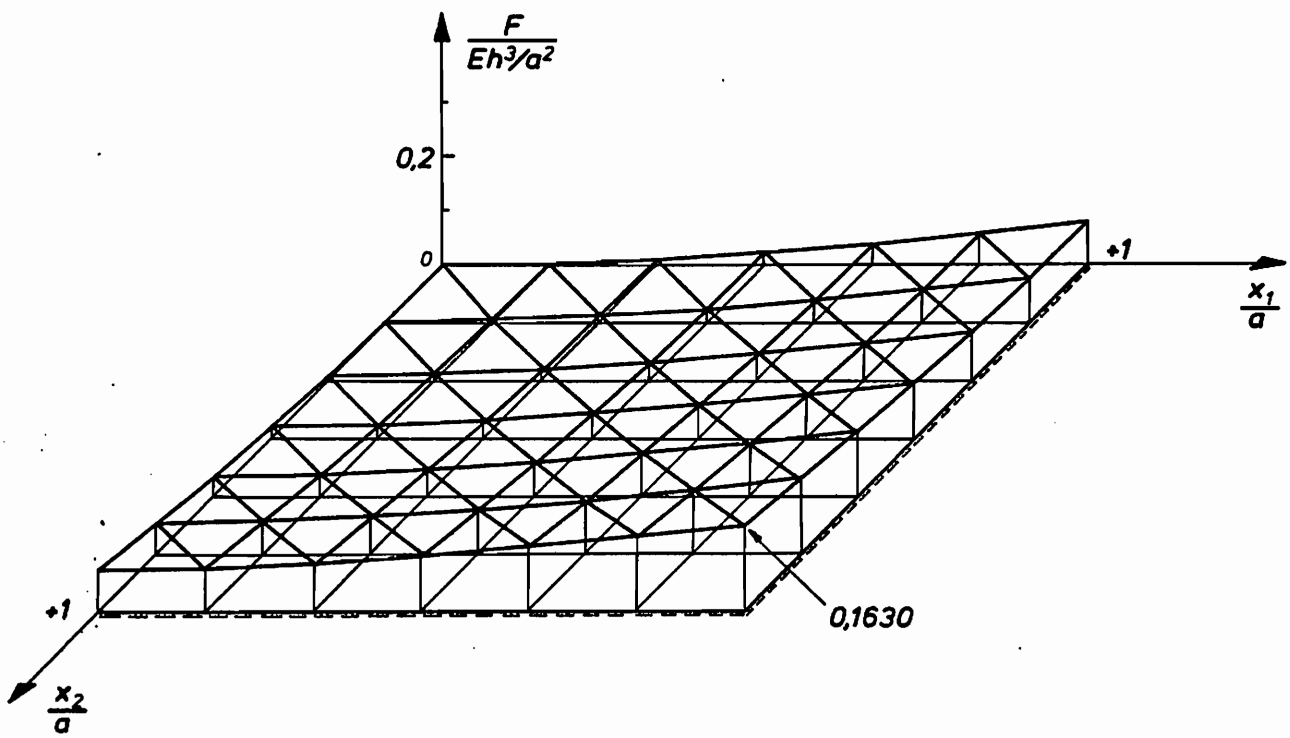

Fig. 28: Airy's stress function

EXAMPLE 8: $\frac{p^*}{Eh^4/a^4} = 1 \quad , \quad \nu = 0,3$

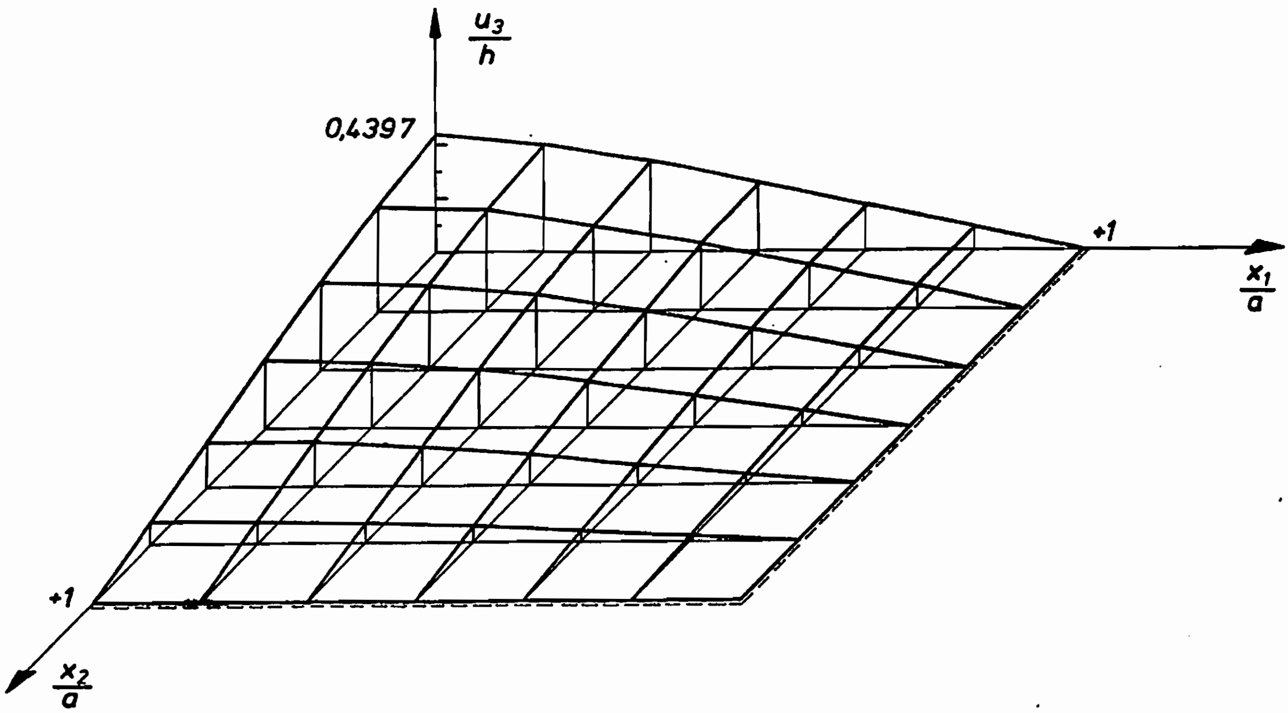

Fig. 29: Deflection

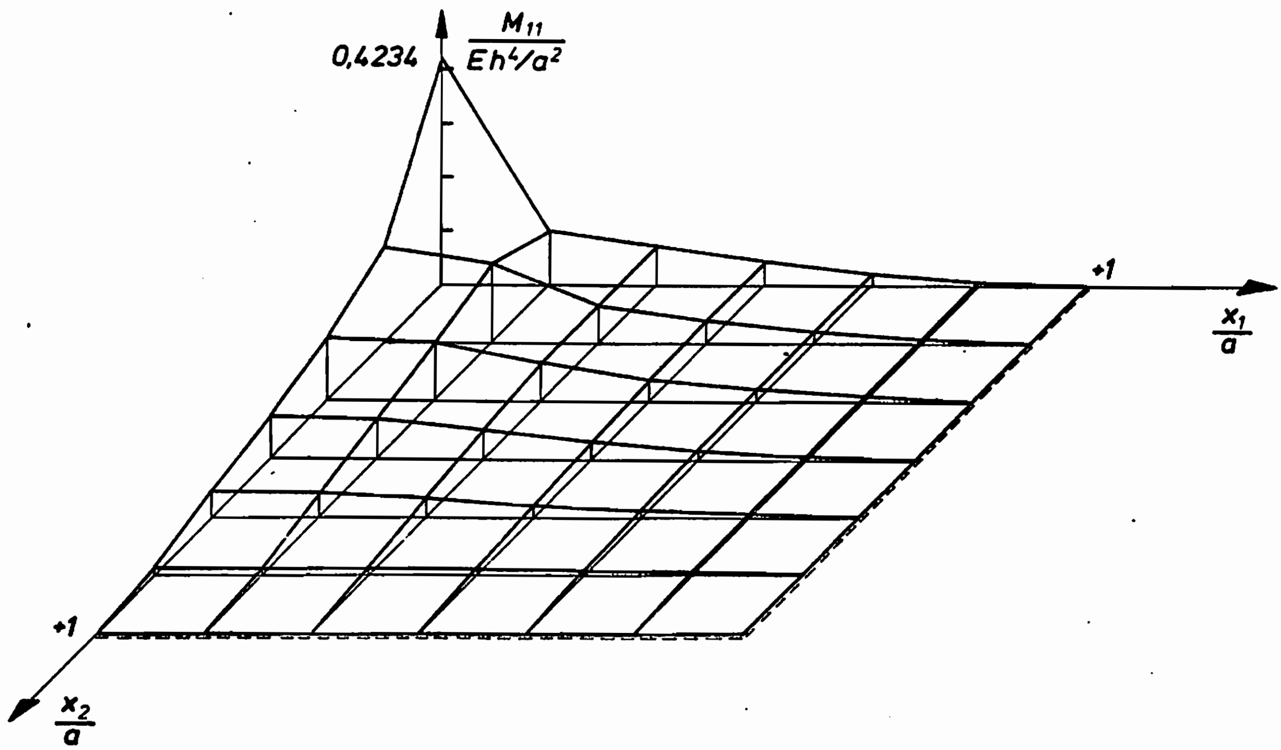

Fig. 30: Bending moment in x_1-direction

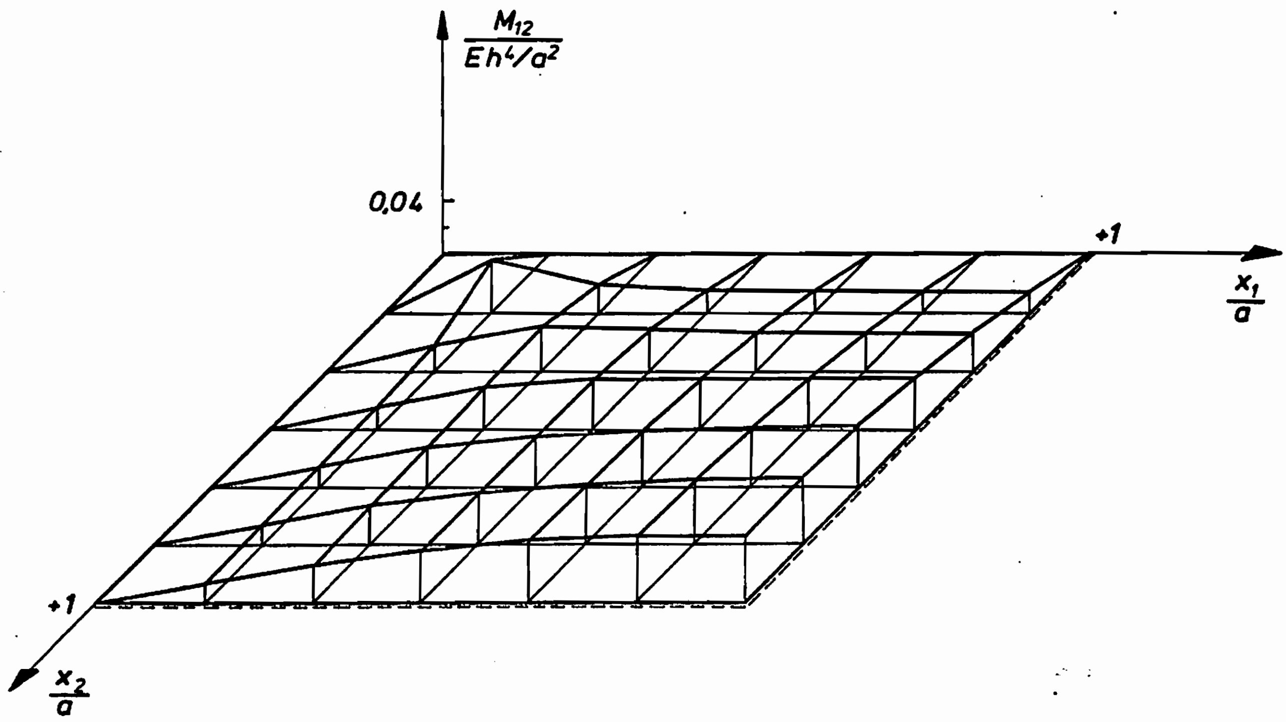

Fig. 31: Twisting moment

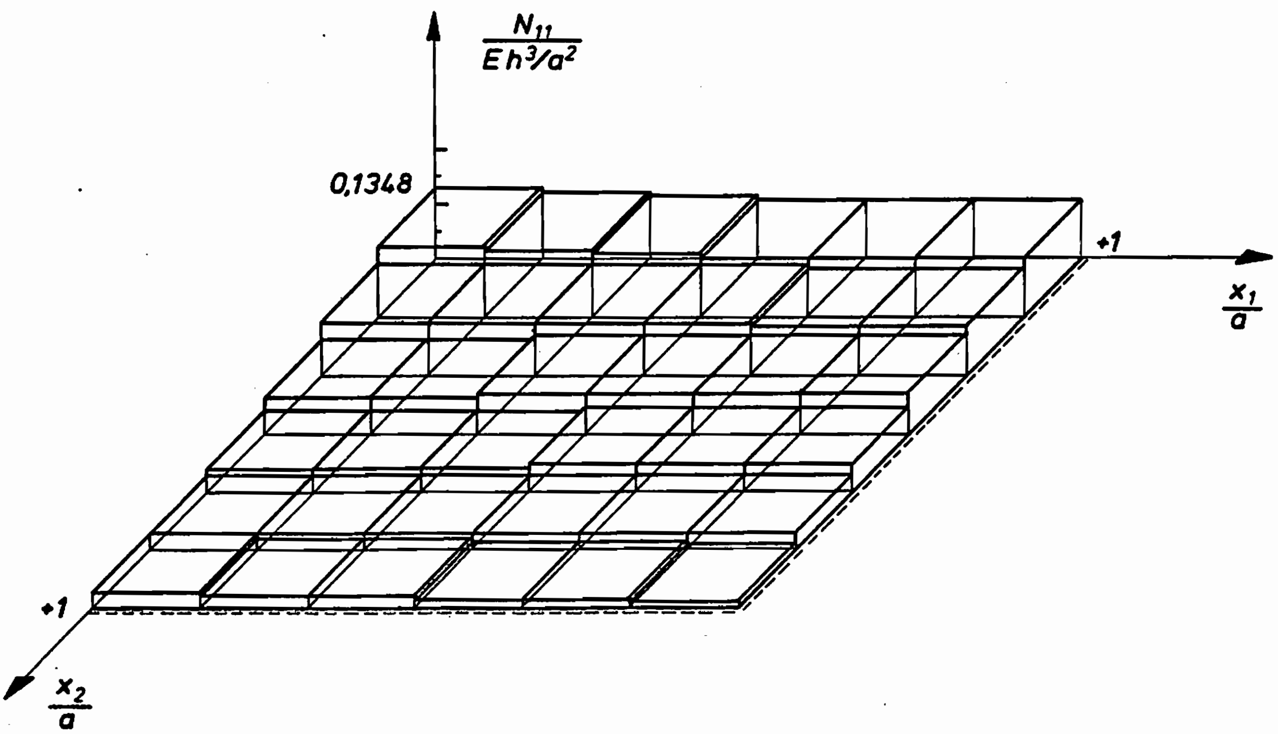

Fig. 32; Normal stresses in x_1-direction

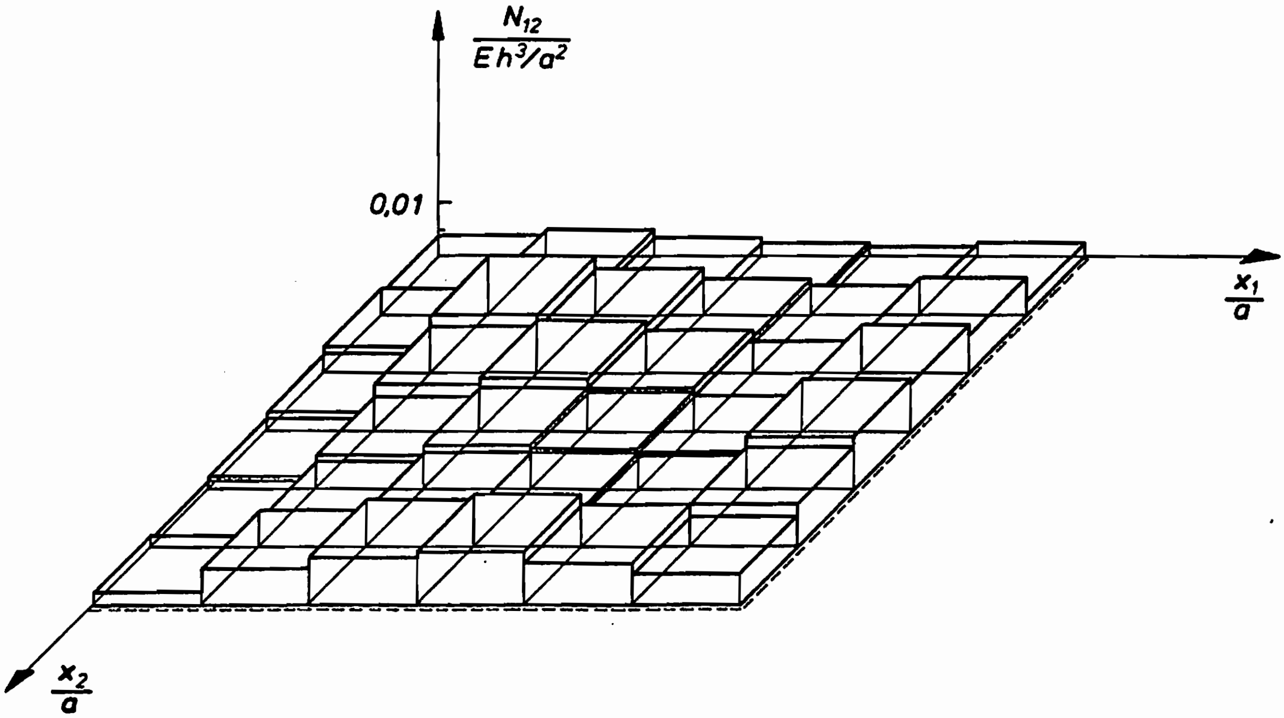

Fig. 33: Shear stresses

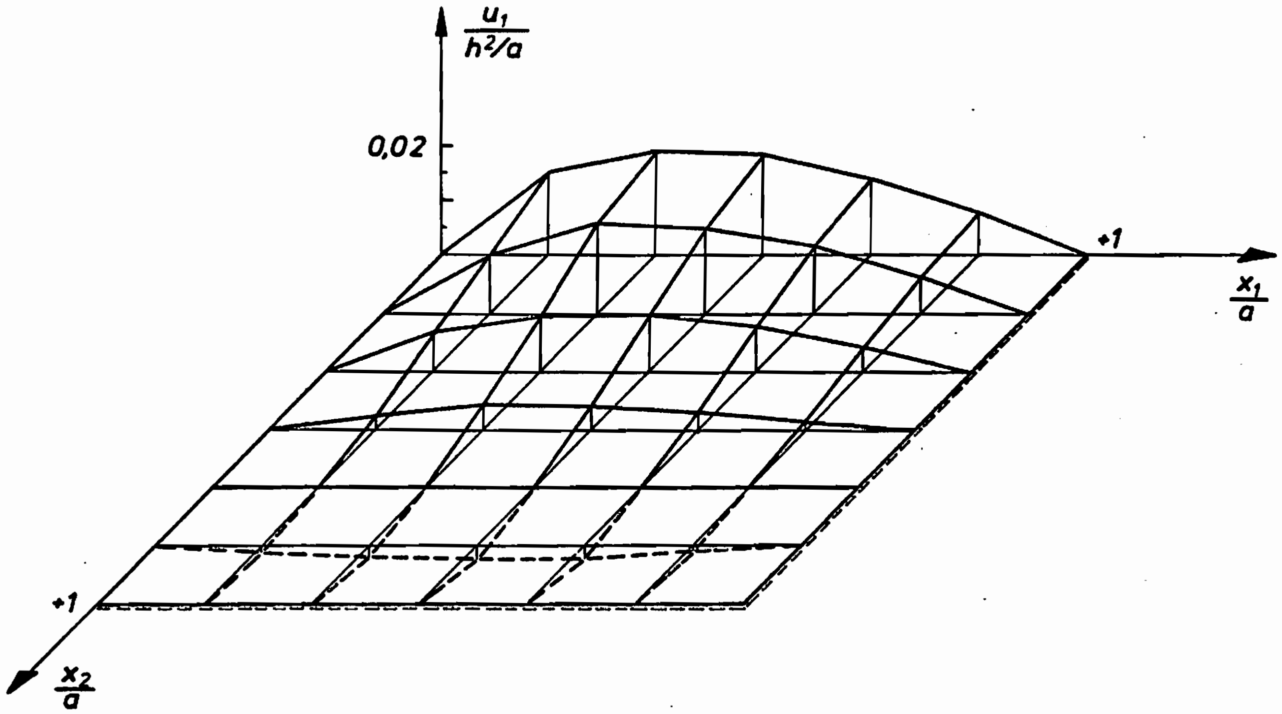

Fig. 34: In-plane displacements

EXAMPLE 9: $\frac{p^*}{Eh^4/a^4} = 1$, $\nu = 0,3$

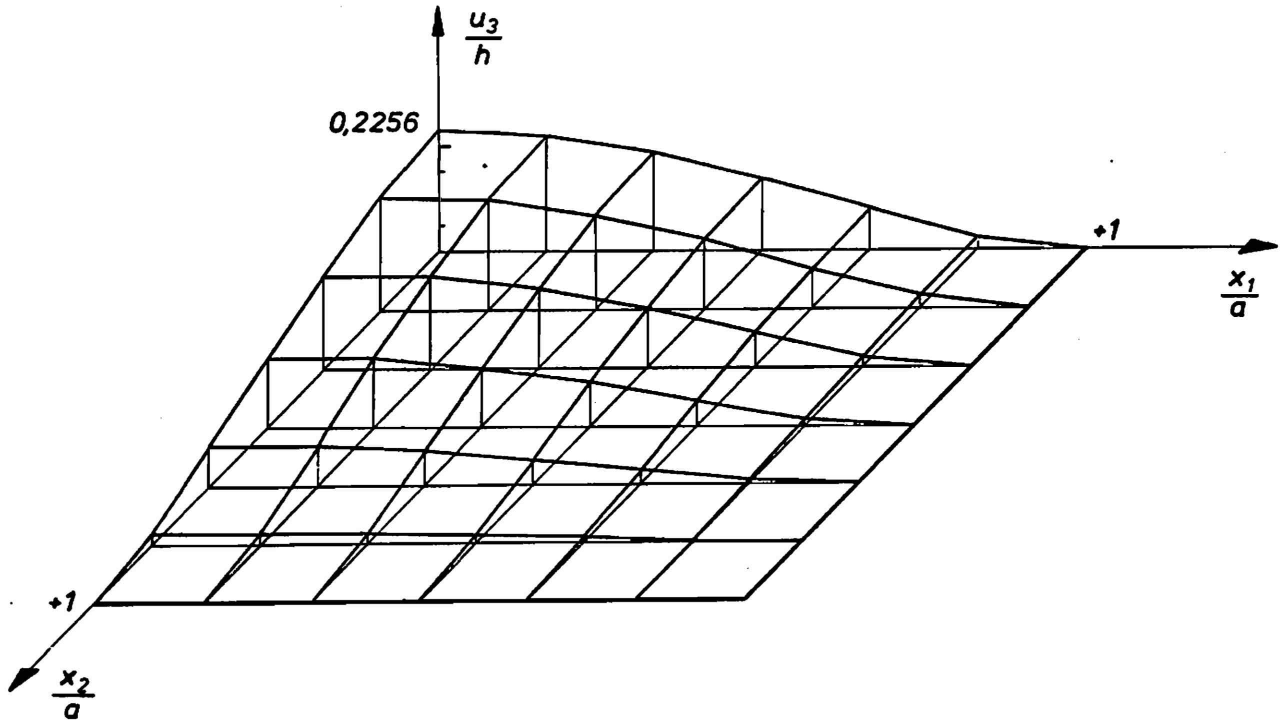

Fig. 35: Deflection

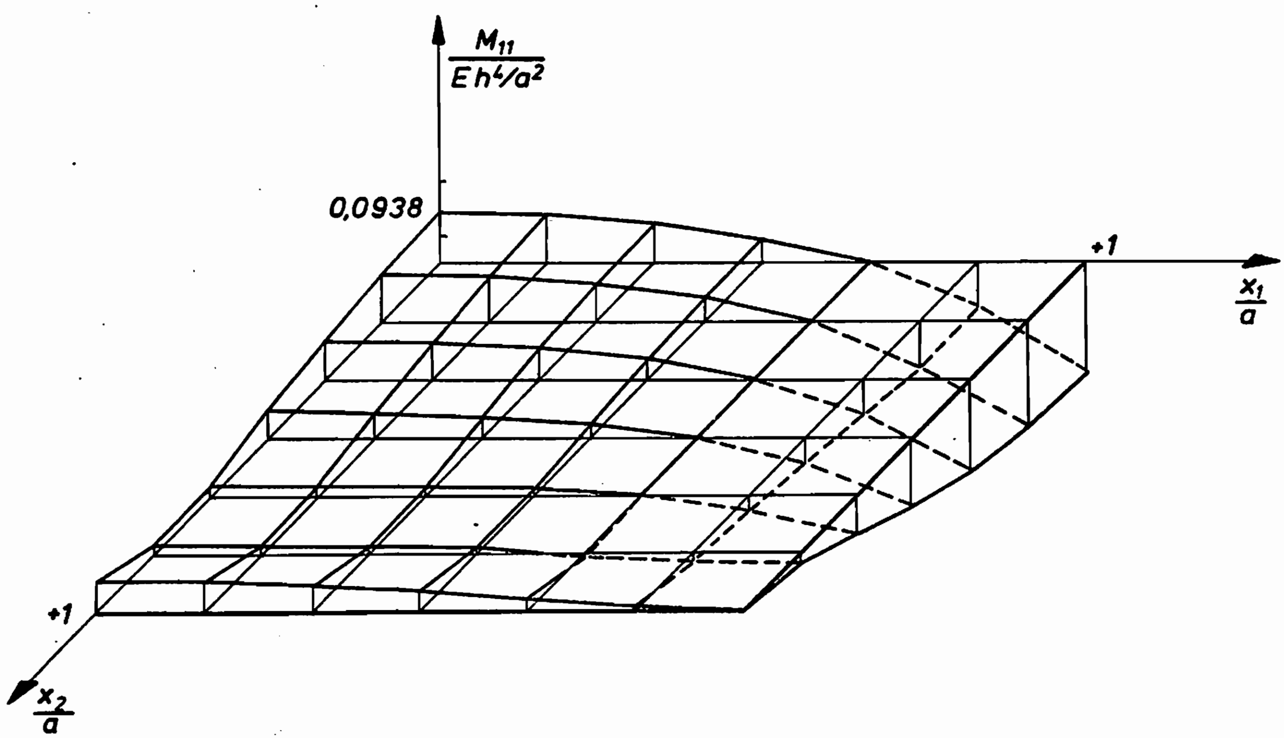

Fig. 36: Bending moment in x_1-direction

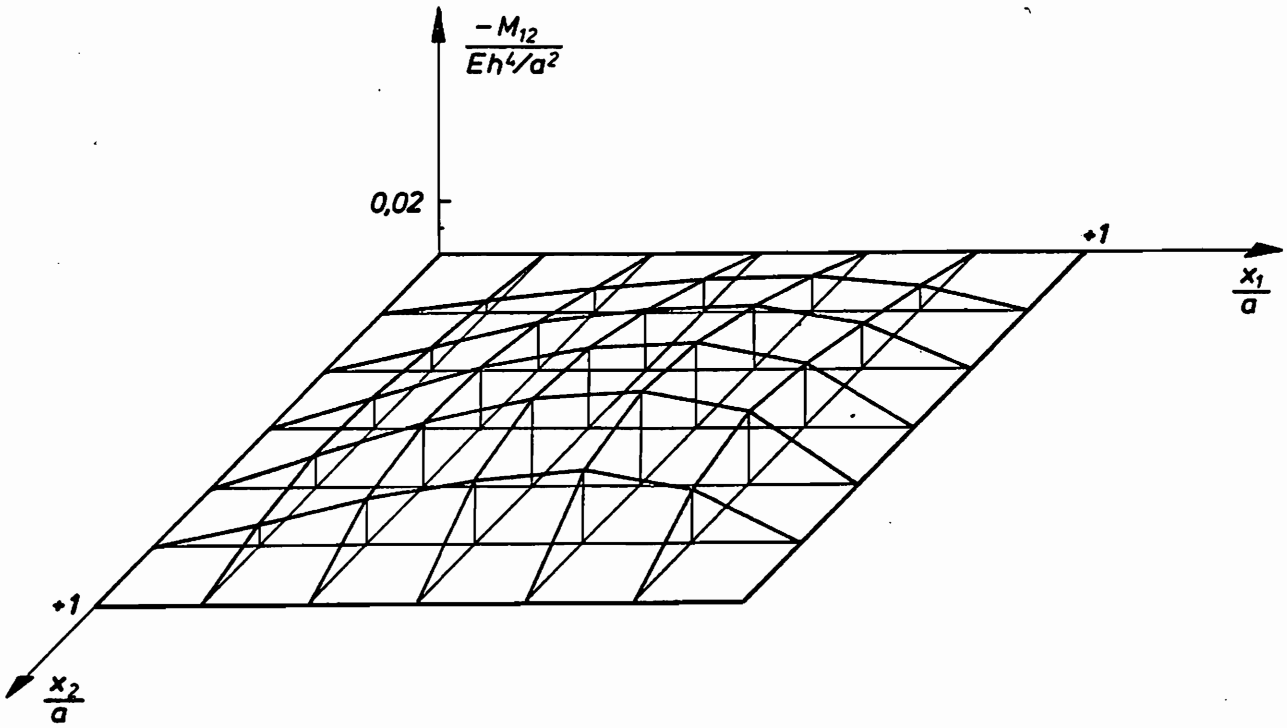

Fig. 37: Twisting moment

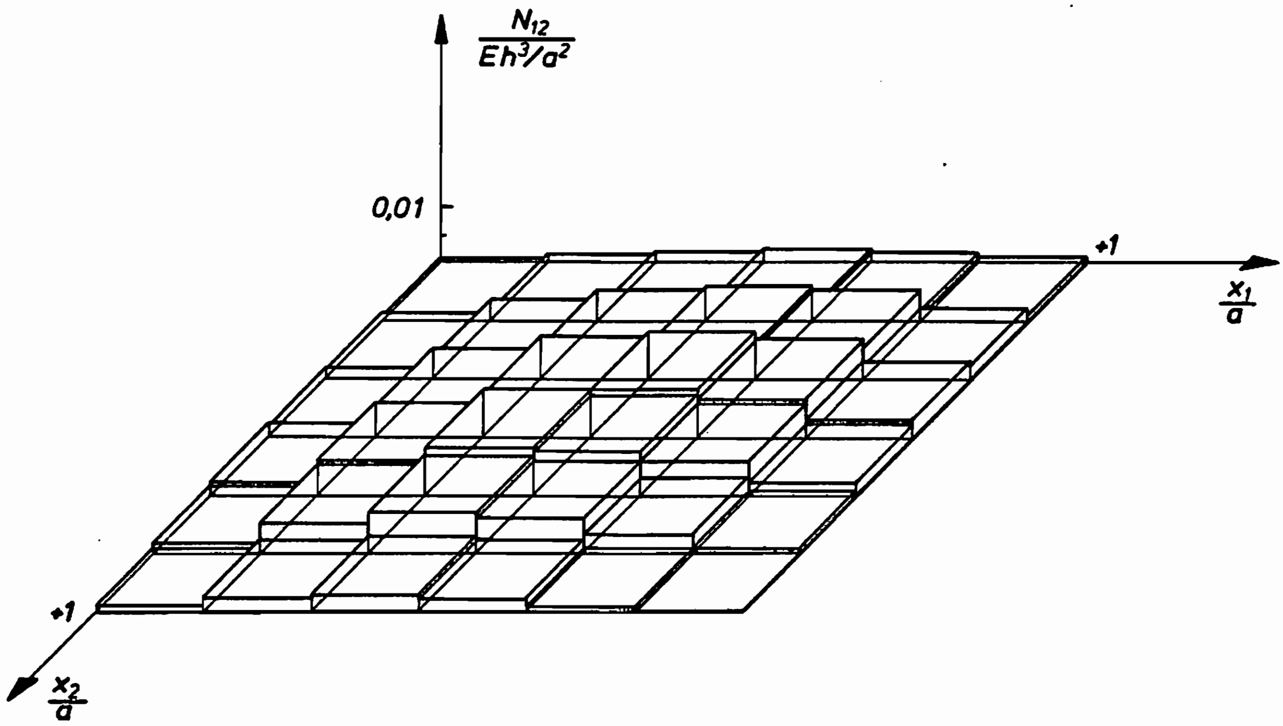

Fig. 38: Shear stresses

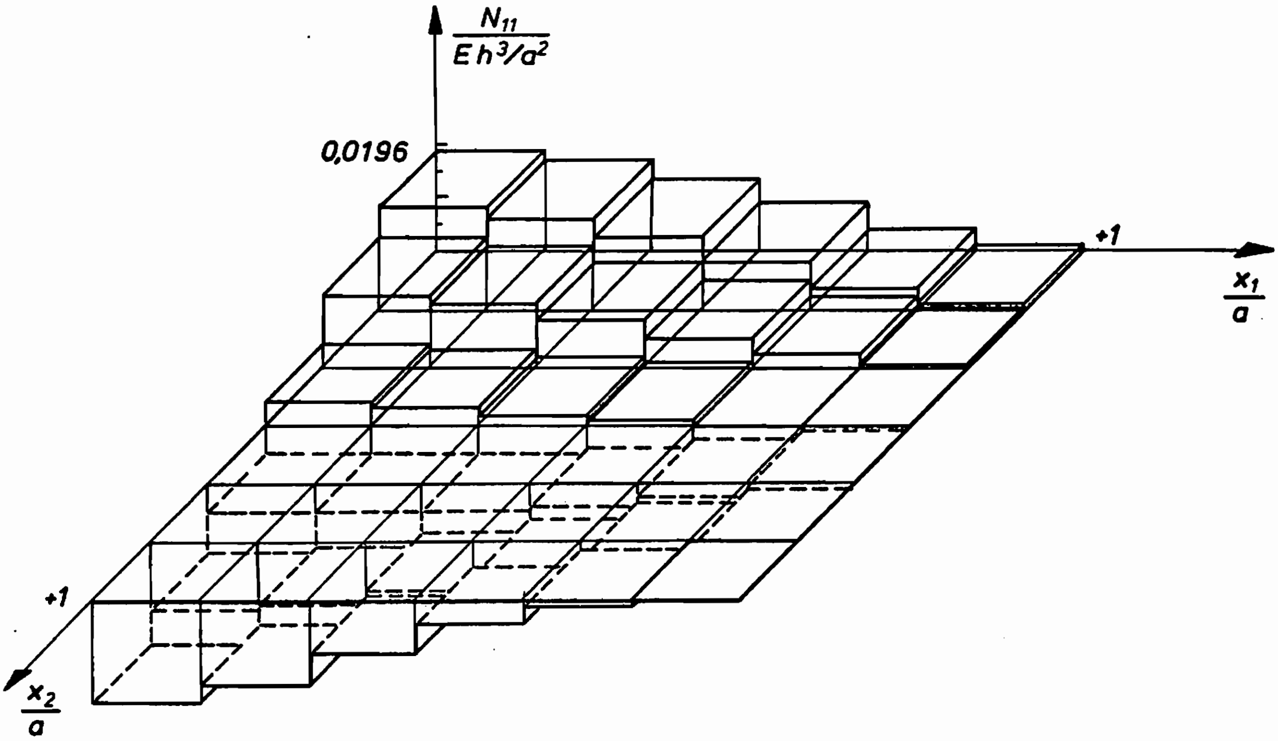

Fig. 39: Normal stresses in x_1-direction

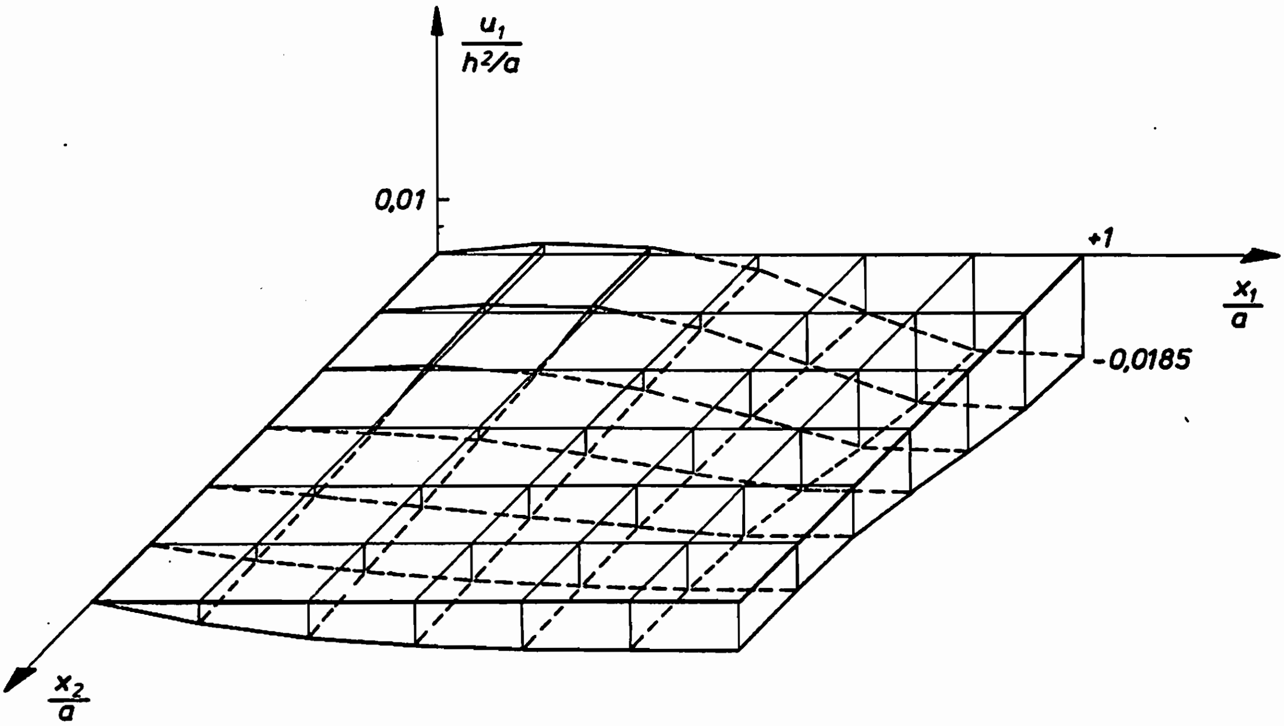

Fig. 40: In-plane displacements in x_1-direction

Incremental calculation of example 2: (s. chapter 8)

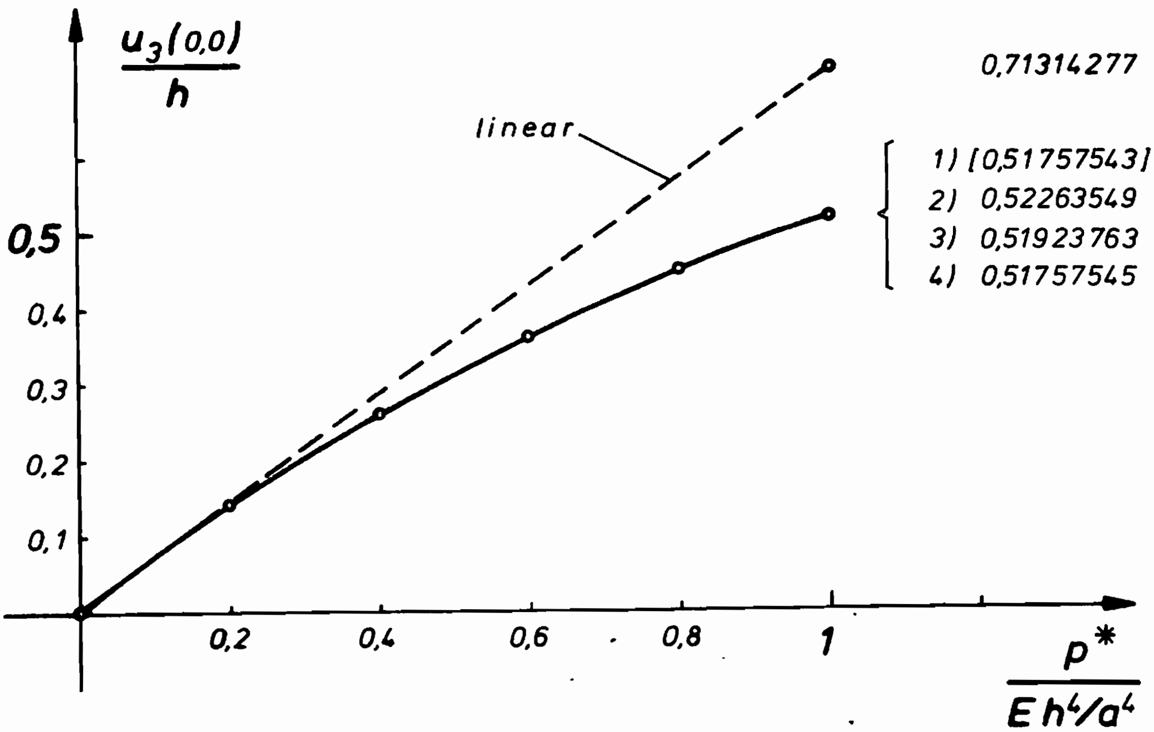

Fig. 41: Load-displacement-curve of the midpoint deflection

1. Newton-Raphson procedure
2. Incrementally without iteration
3. Incrementally with iteration by the system matrix of the system of equations
4. Incrementally with iteration by the right side of the system of equations

11. CONCLUDING REMARKS

In order to gain experience about numerical efficiency of the functionals proposed in chapter 8 and 9, it was reasonable to refer the whole lot of examples to one case of supported plates. The choice was made for the simply supported plate with unmovable edges, because boundary conditions are very simple in such case. In consequence of the regular element division the finite transformation of the functionals is free of coordinates. That is why the input of all data was quite simple. Moreover, all matrices could be calculated by hand before digital calculation set on. As usual, the relation between local and global unknowns was performed by incidence matrices and only after generating the systems of equations, attention was payed to the homogeneous boundary conditions. With respect to the nonlinear problems,this was done by deleting appropriate rows and identifying the corresponding unknowns with zero. After that, the remaining equations and unknowns have been rearranged by another numbering. To the contrary, the order of linear systems of equations remained unaffected. All elements of rows and columns, coming into question from boundary conditions, were set to zero, whereas corresponding diagonal elements were occupied by one. The numbering of global unknowns took place in such way that a bandwidth of the system of equations has been generated small as possible. Nevertheless, for the sake of simplicity, its utilization was renounced and instead of Cholesky's procedure for solving the linear systems of equations the elimination process of Gauß was applied.

It was to be expected that Newton-Raphson's procedure needed more computational storage than the algorithm of incremental calculation. On the other hand, the need for computational time of both procedures was quite the contrary. An immediate calculation of the nonlinear equations took about three to five minutes unlike the incremental method, for which considereably more time was necessary. For instance, the incremental calculation of example 2 ran up to 50 minutes by iterating the system matrix, and to 35 minutes by iterating the right side of the system of equations. Each time, five load increments from null to one had been chosen. Als expected, the number of iterations for both iteration methods were different. An illustration of these facts can be found e. g. in [46]. In summary, there were 31 and 22 iterations respectively for the five load increments mentioned above to be necessary for an accuracy indicated by (9.9).

Of course, it would have been possible, to optimize the functionals by other shape functions too, but it was the intention of the present work to keep numerical efforts small. So, there are still further possibilities to choose shape functions of higher order degrees. By the way, in such cases it might be calculated with a lower number of elements. Since all boundary and interelement boundary conditions were satisfied in functional I_{R1}, its results are obviously the best. This could also be true for functional I_{H1}, but here a lot of numerical work would be necessary for computing the function of energy density, and this is not recommendable to do for shape functions of higher order degrees. Against that, all other functionals are qualified for such shape functions.

Experiences, which have been made during numerical calculations have shown that functionals with jump terms are very stable numerically. Just slight errors in the functionals I_{R1} and I_{H1} broke off digital calcualtions unlike in other functionals, for which computational results still had been obtained. This was extremely helpful in searching mistakes. Functional I_{R2} turned out to be the best with respect to practical operation. It is convenient for writing computer programs and numerical results are assured to be as good as those from I_{R1}. For this reason, functional I_{R2} was chosen for the incremental method. The results, which have been obtained in this manner, are identical with those of Newton-Raphson's method. This is recognizable already by the values of the midpoint deflection plotted in figure 41.

At the end of comments, the other values of the midpoint deflection may be listed, which have been gained by application of the Newton-Raphson procedure to all other functionals referring to the simply supported plate:

Examples	u_3/h
1	0,51705263
2	0,51757543
3	0,51788251
4	0,52165107
5	0,51518353
6	0,51600689
7	0,52005804

APPENDIX A

This first chapter A of the appendix serves to recall elementary relations of tensor calculus on curved surfaces. The special case of plane surfaces becomes evident, if the tensor of curvature $b_{\alpha\beta}$ is set to zero in all considerations treated completely general.

A1 Geometrical relations on surfaces of any curvature

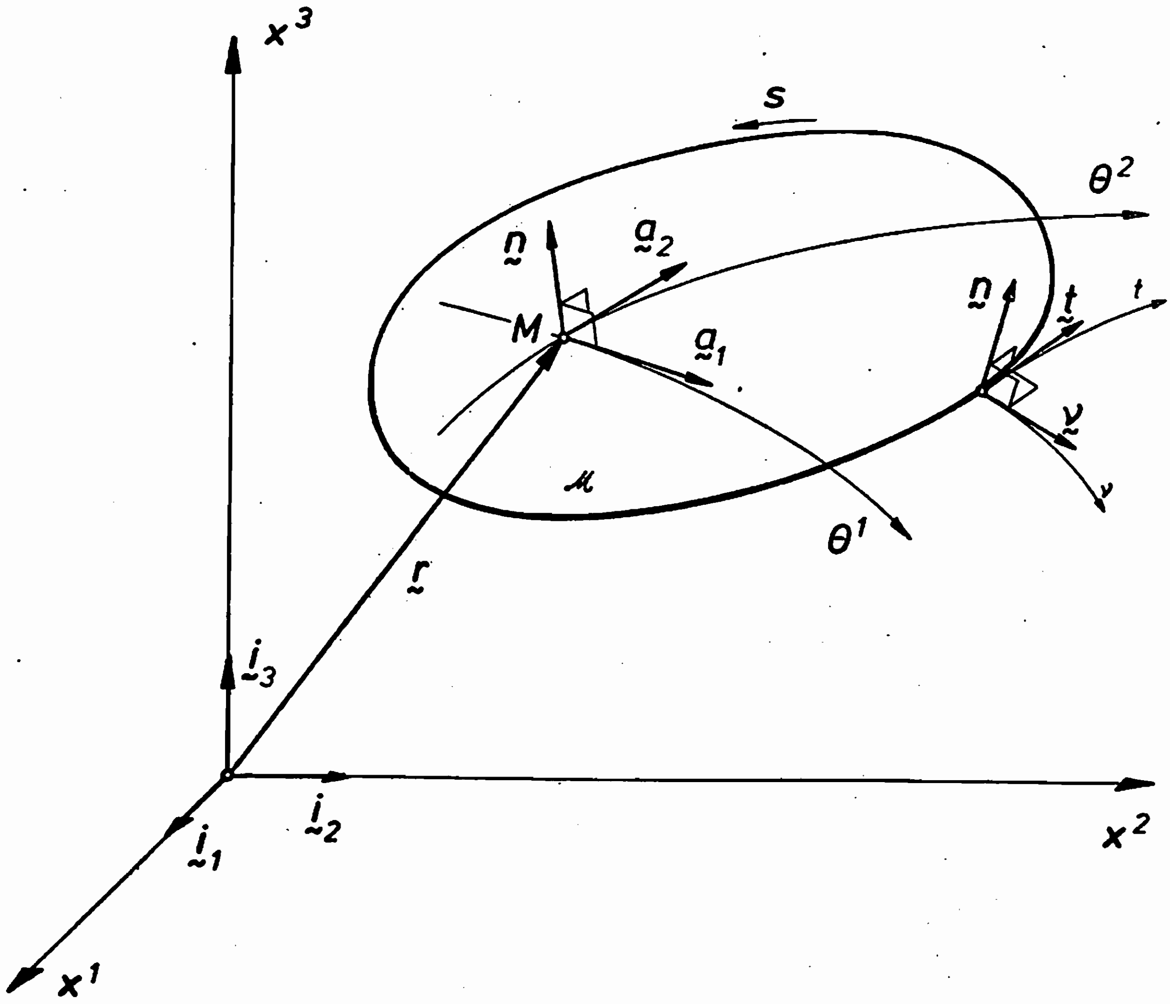

Fig. A1

Any given surface M in threedimensional Eukledean space can be defined by the vector function (s. figure A1)

$$\underset{\sim}{r} = \overrightarrow{OM} = f(\Theta^{\alpha}) = \sum_{\kappa=1}^{3} f^{\kappa}(\Theta^{\alpha})\,\underset{\sim}{i}_{\kappa} = x^{\kappa}\,\underset{\sim}{i}_{\kappa}. \tag{A1.1}$$

By partial differentiation of $\underset{\sim}{r}$ with respect to x^{α} and θ^{α} respectively *covariant base vectors* are defined in both coordinate systems.

$$\frac{\partial \underset{\sim}{r}}{\partial x^{\kappa}} = \underset{\sim}{r}_{,\kappa} = \underset{\sim}{i}_{\kappa} \tag{A1.2}$$

$$\frac{\partial \underset{\sim}{r}}{\partial \Theta^{\alpha}} = \underset{\sim}{r}_{,\alpha} = \frac{\partial x^{\kappa}}{\partial \Theta^{\alpha}}\,\underset{\sim}{i}_{\kappa} = \underset{\sim}{a}_{\alpha} \tag{A1.3}$$

With *Kronecker's symbol* δ_{α}^{β} *contravariant base vectors* $\underset{\sim}{a}^{\beta}$ are introduced into the system of curvilinear coordinates according to the definition

$$\underset{\sim}{a}_{\alpha}\cdot\underset{\sim}{a}^{\beta} = \delta_{\alpha}^{\beta} = \begin{cases} & \text{für } \alpha = \beta \\ & \\ & \text{für } \alpha \neq \beta\,. \end{cases} \tag{A1.4}$$

In cartesian coordinates covariant and contravariant base vectors coincide. Because

$$\frac{\partial \Theta^{\alpha}}{\partial \Theta^{\beta}} = \frac{\partial \Theta^{\alpha}}{\partial x^{\kappa}}\,\frac{\partial x^{\kappa}}{\partial \Theta^{\beta}} = \delta_{\beta}^{\alpha}$$

holds in case of admissible coordinate transformations, with respect to (A1.3) a corresponding relation can be given for the contravariant basis too.

$$\frac{\partial \Theta^{\alpha}}{\partial x^{\kappa}}\,\underset{\sim}{i}^{\kappa} = \underset{\sim}{a}^{\alpha} \tag{A1.5}$$

Both vector bases may be represented mutually as linear combinations by the formulae

$$\left.\begin{aligned} \underset{\sim}{a}_{\alpha} &= a_{\alpha\mu}\,\underset{\sim}{a}^{\mu} &\Longrightarrow\quad a_{\alpha\mu} &= \underset{\sim}{a}_{\alpha}\cdot\underset{\sim}{a}_{\mu} \\ \underset{\sim}{a}^{\alpha} &= a^{\alpha\mu}\,\underset{\sim}{a}_{\mu} &\Longrightarrow\quad a^{\alpha\mu} &= \underset{\sim}{a}^{\alpha}\cdot\underset{\sim}{a}^{\mu} \end{aligned}\right\} \tag{A1.6}$$

where the linear factors $a_{\alpha\mu}$ and $a^{\alpha\mu}$ are called *covariant* and *contravariant metric tensor components.*

The orthogonality relation

$$a_{\alpha\mu}\, a^{\beta\mu} = \delta_\alpha^\beta$$

results from (A1.4) by use of (A1.6).

If (A1.6) is substituted into the basis of any tensor $\underset{\sim}{T}$ of n-th order, for example

$$\underset{\sim}{T} = T^{\alpha\beta\dots}{}_{\dots\gamma\delta}\, \underset{\sim}{a}_\alpha \otimes \underset{\sim}{a}_\beta \otimes \cdots \otimes \underset{\sim}{a}^\gamma \otimes \underset{\sim}{a}^\delta$$

then it shows, that metric tensor components raise up and down indices of the given tensor, as for instance

$$a_{\alpha\mu}\, a^{\gamma\nu}\, T^{\alpha\beta\dots}{}_{\dots\gamma\delta} = T_\mu{}^{\cdot\beta\dots\nu}{}_{\dots\dots\delta} \,. \tag{A1.7}$$

By definition of the *permutation tensor* $\underset{\sim}{\epsilon}$ the base vectors are related to the *unit normal* $\underset{\sim}{n}$ as follows:

$$\left.\begin{aligned} \epsilon_{\alpha\beta} &= (\underset{\sim}{a}_\alpha \times \underset{\sim}{a}_\beta)\cdot \underset{\sim}{n} \\ \epsilon^{\alpha\beta} &= (\underset{\sim}{a}^\alpha \times \underset{\sim}{a}^\beta)\cdot \underset{\sim}{n} \end{aligned}\right\} \tag{A1.8}$$

With a = det $(a_{\alpha\beta})$ this is equivalent to:

$$\epsilon_{\alpha\beta} = \sqrt{a} \begin{cases} 1 & \alpha = 1,\ \beta = 2 \\ 0 & \alpha = \beta \\ -1 & \alpha = 2,\ \beta = 1 \end{cases}$$

$$\epsilon^{\alpha\beta} = \frac{1}{\sqrt{a}} \begin{cases} 1 & \alpha = 1,\ \beta = 2 \\ 0 & \alpha = \beta \\ -1 & \alpha = 2,\ \beta = 1 \end{cases}$$

As is known, the δ-tensor of fourth order is originated by tensorial multiplication of both permutation tensors.

$$\epsilon_{\alpha\beta}\, \epsilon^{\lambda\mu} = \delta^{\lambda\mu}_{\alpha\beta} = \delta_\alpha^\lambda \delta_\beta^\mu - \delta_\beta^\lambda \delta_\alpha^\mu \tag{A1.9}$$

Figure A1 shows a triad of orthonormal vectors $\underset{\sim}{\nu}$, $\underset{\sim}{t}$, $\underset{\sim}{n}$, which move along the boundary curve C. The *unit normal* $\underset{\sim}{n}$ is declared already by (A1.8). The *unit tangent* $\underset{\sim}{t}$ and the *outward unit normal* $\underset{\sim}{\nu}$ are defined to be

$$\left.\begin{aligned}\underset{\sim}{t} &= \frac{d\underset{\sim}{r}}{ds} = \underset{\sim}{r}_{,\alpha}\,\frac{\partial\Theta^{\alpha}}{\partial s} = \underset{\sim}{a}_{\alpha}\,t^{\alpha}\\ \underset{\sim}{\nu} &= \frac{d\underset{\sim}{r}}{d\nu} = \underset{\sim}{r}_{,\alpha}\,\frac{\partial\Theta^{\alpha}}{\partial\nu} = \underset{\sim}{a}_{\alpha}\,\nu^{\alpha}\end{aligned}\right\}. \qquad \text{(A1.10)}$$

Because of

$$\underset{\sim}{t}^{2} = \underset{\sim}{\nu}^{2} = 1$$

$$\underset{\sim}{\nu}\cdot\underset{\sim}{t} = 0$$

the following orthogonality relations must be valid.

$$\left.\begin{aligned}t^{\alpha}t_{\alpha} &= 1\\ \nu^{\alpha}\nu_{\alpha} &= 1\\ t^{\alpha}\nu_{\alpha} &= 0\end{aligned}\right\} \qquad \text{(A1.11)}$$

These relations are used in the definition of $\underset{\sim}{n}$.

$$\underset{\sim}{n} = \underset{\sim}{\nu}\times\underset{\sim}{t} = \nu_{\alpha}\,\underset{\sim}{a}^{\alpha}\times t_{\beta}\,\underset{\sim}{a}^{\beta} = \epsilon^{\alpha\beta}\,\nu_{\alpha}\,t_{\beta}\,\underset{\sim}{n}$$

At first, this identity leads to the equations

$$\epsilon^{\alpha\beta}\,\nu_{\alpha}\,t_{\beta} = 1$$

$$\epsilon_{\alpha\beta}\,\nu^{\alpha}t^{\beta} = 1\,.$$

Due to (A1.11), a representation of the components is found.

$$\left.\begin{aligned}\nu^{\alpha} &= \epsilon^{\alpha\beta}\,t_{\beta}\,, & t^{\alpha} &= \epsilon^{\beta\alpha}\,\nu_{\beta}\\ \nu_{\alpha} &= \epsilon_{\alpha\beta}\,t^{\beta}\,, & t_{\alpha} &= \epsilon_{\beta\alpha}\,\nu^{\beta}\end{aligned}\right\} \qquad \text{(A1.12)}$$

A2 The transformation of tensors

The definition of any tensor is based on its *behaviour of transformation*, if a new coordinate system Θ'^{α} is introduced. It is possible to express new and old coordinates by each other.

$$\left.\begin{aligned}\Theta'^{\alpha} &= \Theta'^{\alpha}(\Theta^{\beta})\\ \Theta^{\alpha} &= \Theta^{\alpha}(\Theta'^{\beta})\end{aligned}\right\} \qquad \text{(A2.1)}$$

This is only true, if the *Jacobian determinant*

$$\mathcal{J} = \det\left(\frac{\partial(\Theta^1,\Theta^2)}{\partial(\Theta'^1,\Theta'^2)}\right) \tag{A2.2}$$

is neither zero nor infinite.

Characteristic for admissible coordinate transformations is the validity of

$$\left.\begin{aligned} \frac{\partial\Theta^\beta}{\partial\Theta^\alpha} &= \frac{\partial\Theta'^\mu}{\partial\Theta^\alpha}\frac{\partial\Theta^\beta}{\partial\Theta'^\mu} = \delta^\beta_\alpha \\ \frac{\partial\Theta'^\beta}{\partial\Theta'^\alpha} &= \frac{\partial\Theta^\mu}{\partial\Theta'^\alpha}\frac{\partial\Theta'^\beta}{\partial\Theta^\mu} = \delta^\beta_\alpha \end{aligned}\right\} . \tag{A2.3}$$

By use of these orthogonalities in conjunction with

$$\left.\begin{aligned} \underset{\sim}{a}_\alpha &= \frac{\partial\Theta'^\beta}{\partial\Theta^\alpha}\underset{\sim}{a}'_\beta \\ \underset{\sim}{a}'_\alpha &= \frac{\partial\Theta^\beta}{\partial\Theta'^\alpha}\underset{\sim}{a}_\beta \end{aligned}\right\} \tag{A2.4}$$

transformation rules for contravariant base vectors are obtained.

$$\left.\begin{aligned} \underset{\sim}{a}^\alpha &= \frac{\partial\Theta^\alpha}{\partial\Theta'^\beta}\underset{\sim}{a}'^\beta \\ \underset{\sim}{a}'^\alpha &= \frac{\partial\Theta'^\alpha}{\partial\Theta^\beta}\underset{\sim}{a}^\beta \end{aligned}\right\} \tag{A2.5}$$

The tensor $\underset{\sim}{T}$ may be referred to old and new base vectors.

$$\begin{aligned} \underset{\sim}{T} &= T^{\alpha\beta\dots}{}_{\dots\gamma\delta}\,\underset{\sim}{a}_\alpha \otimes \underset{\sim}{a}_\beta \otimes \dots\dots \otimes \underset{\sim}{a}^\gamma \otimes \underset{\sim}{a}^\delta \\ &= T'^{\mu\nu\dots}{}_{\dots\varphi\psi}\,\underset{\sim}{a}'_\mu \otimes \underset{\sim}{a}'_\nu \otimes \dots\dots \otimes \underset{\sim}{a}'^\varphi \otimes \underset{\sim}{a}'^\psi \end{aligned}$$

Now, the application of (A2.4) and (A2.5) yields finally the transformation rules for tensor components.

$$\left.\begin{aligned} T'^{\mu\nu\dots}{}_{\dots\varphi\psi} &= \frac{\partial\Theta'^\mu}{\partial\Theta^\alpha}\frac{\partial\Theta'^\nu}{\partial\Theta^\beta}\dots\frac{\partial\Theta^\gamma}{\partial\Theta'^\varphi}\frac{\partial\Theta^\delta}{\partial\Theta'^\psi}\,T^{\alpha\beta\dots}{}_{\dots\gamma\delta} \\ T^{\alpha\beta\dots}{}_{\dots\gamma\delta} &= \frac{\partial\Theta^\alpha}{\partial\Theta'^\mu}\frac{\partial\Theta^\beta}{\partial\Theta'^\nu}\dots\frac{\partial\Theta'^\varphi}{\partial\Theta^\gamma}\frac{\partial\Theta'^\psi}{\partial\Theta^\delta}\,T'^{\mu\nu\dots}{}_{\dots\varphi\psi} \end{aligned}\right\} \tag{A2.6}$$

A3 The covariant derivative

Partial derivatives of the base vectors $\underset{\sim}{a}_\alpha$, $\underset{\sim}{a}^\alpha$ and $\underset{\sim}{n}$ respectively along parameter lines generate new vectors, which can be referred to the original basis again. For any surface, embedded in threedimensional Euclidean space, these equations of derivatives are well-known and they are called according to *Gauß-Weingarten* [48].

$$\left.\begin{aligned} \underset{\sim}{a}_{\alpha,\beta} &= \Gamma^{\gamma}_{\alpha\beta}\,\underset{\sim}{a}_\gamma + b_{\alpha\beta}\,\underset{\sim}{n} \\ \underset{\sim}{a}^{\alpha}{}_{,\beta} &= -\Gamma^{\alpha}_{\beta\gamma}\,\underset{\sim}{a}^\gamma + b^{\alpha}_{\beta}\,\underset{\sim}{n} \\ \underset{\sim}{n}_{,\alpha} &= -\,b^{\beta}_{\alpha}\,\underset{\sim}{a}_\beta \end{aligned}\right\} \qquad (A3.1)$$

Herein, *Christoffel's symbol* $\Gamma^{\gamma}_{\alpha\beta}$ is not a tensor, since it does not satisfy the transformation rule of a third-ordered tensor according to (A2.6). The equation of definition for the Christoffel symbol reads:

$$\Gamma^{\gamma}_{\alpha\beta} = \tfrac{1}{2}\,a^{\gamma\lambda}\left(a_{\lambda\beta,\alpha} + a_{\alpha\lambda,\beta} - a_{\beta\alpha,\lambda}\right) \qquad (A3.2)$$

Definite explanations about curvature properties of any surface are obtained by the *tensor of curvature* $b_{\alpha\beta}$ having come into appearence in (A3.1).

$$b_{\alpha\beta} = -\underset{\sim}{n}_{,\alpha}\cdot\underset{\sim}{a}_\beta = -\underset{\sim}{n}_{,\beta}\cdot\underset{\sim}{a}_\alpha = \underset{\sim}{n}\cdot\underset{\sim}{a}_{\alpha,\beta} = \underset{\sim}{n}\cdot\underset{\sim}{a}_{\beta,\alpha} \qquad (A3.3)$$

By means of the equations of Gauß-Weingarten (A3.1) it is possible to differentiate a tensor of any order with respect to Gaussian parameter lines. In case of space vector it follows

$$\left.\begin{aligned} \underset{\sim}{u} &= u_\alpha\,\underset{\sim}{a}^\alpha + u_3\,\underset{\sim}{n} \\ \underset{\sim}{u}_{,\beta} &= \left(u_\alpha|_\beta - b_{\alpha\beta}\,u_3\right)\underset{\sim}{a}^\alpha + u_3|_\beta\,\underset{\sim}{n} \end{aligned}\right\}. \qquad (A3.4)$$

Herein,

$$u_\alpha|_\beta = u_{\alpha,\beta} + \Gamma^{\gamma}_{\alpha\beta}\,u_\gamma \qquad (A3.5)$$

$$u_3|_\beta = u_{3,\beta} + b^{\alpha}_{\beta}\,u_\alpha \qquad (A3.6)$$

signify covariant derivatives of u_α and u_3.

They are tensors as it can be verified by proving their behaviour of transformation.

If u_α is a surface vector, then u_3 vanishes in (A3.4). In special case of a plane surface, there is no difference between partial and covariant derivative of u_3 since $b_{\alpha\beta} = 0$.

$$u_3|_\kappa = u_{3,\kappa} \tag{A3.7}$$

Generally, this is always true for a *scalar function.*

Partial differentiation of a second order tensor turns out to be quite analogous to that of a vector. Again, equations of Gauß-Weingarten are used.

$$\begin{aligned} \underset{\sim}{T}_{,\gamma} &= (T_{\kappa\beta}\, \underset{\sim}{a}^\kappa \otimes \underset{\sim}{a}^\beta)_{,\gamma} \\ &= T_{\kappa\beta}|_\gamma\, \underset{\sim}{a}^\kappa \otimes \underset{\sim}{a}^\beta + b^\kappa_\gamma\, T_{\kappa\beta}\, \underset{\sim}{n} \otimes \underset{\sim}{a}_\alpha + b^\beta_\gamma\, T_{\kappa\beta}\, \underset{\sim}{a}^\kappa \otimes \underset{\sim}{n} \end{aligned} \tag{A.3.8}$$

The covariant derivative of the tensor components

$$T_{\kappa\beta}|_\gamma = T_{\kappa\beta,\gamma} - \Gamma^\kappa_{\kappa\gamma}\, T_{\kappa\beta} - \Gamma^\kappa_{\beta\gamma}\, T_{\kappa\kappa} \tag{A3.9}$$

satisfies the transformation rule of (A2.6) for a third ordered tensor. Further expositions for tensor components of mixed type and for higher ordered tensors may be dropped for the sake of shortness.

Within these facts, a significant role plays the *surface Riemann-Christoffel tensor,* which is related to twice covariant derivatives of a vector by the formula

$$u_\alpha|_{\beta\gamma} - u_\alpha|_{\gamma\beta} = R^{\kappa}_{\cdot\,\alpha\beta\gamma}\, u_\kappa . \tag{A3.10}$$

To its definition the tensor of curvature $b_{\alpha\beta}$ is available, that is

$$R_{\kappa\beta\lambda\mu} = \delta^{\gamma\nu}_{\lambda\mu}\, b_{\kappa\gamma}\, b_{\beta\nu} . \tag{A3.11}$$

By means of the *Gaussian curvature* K another definition is possible.

$$R_{\kappa\beta\lambda\mu} = \epsilon_{\kappa\beta}\, \epsilon_{\lambda\mu}\, K \tag{A3.12}$$

with

$$K = \det(b^\beta_\alpha) = \tfrac{1}{2}\, \delta^{\beta\nu}_{\alpha\mu}\, b^\alpha_\beta\, b^\mu_\nu \tag{A3.13}$$

In Euclidean spaces the sequence of covariant derivatives is interchangeable since Riemann's Christoffel tensor is equal to zero in that case. It shall be recalled that certain derivatives vanish identically:

$$a^{\alpha\beta}/_{\lambda} = a_{\alpha\beta}/_{\lambda} = 0$$

$$\epsilon^{\alpha\beta}/_{\lambda} = \epsilon_{\alpha\beta}/_{\lambda} = 0$$

APPENDIX B

B1 Representative descriptions in the system of the triad

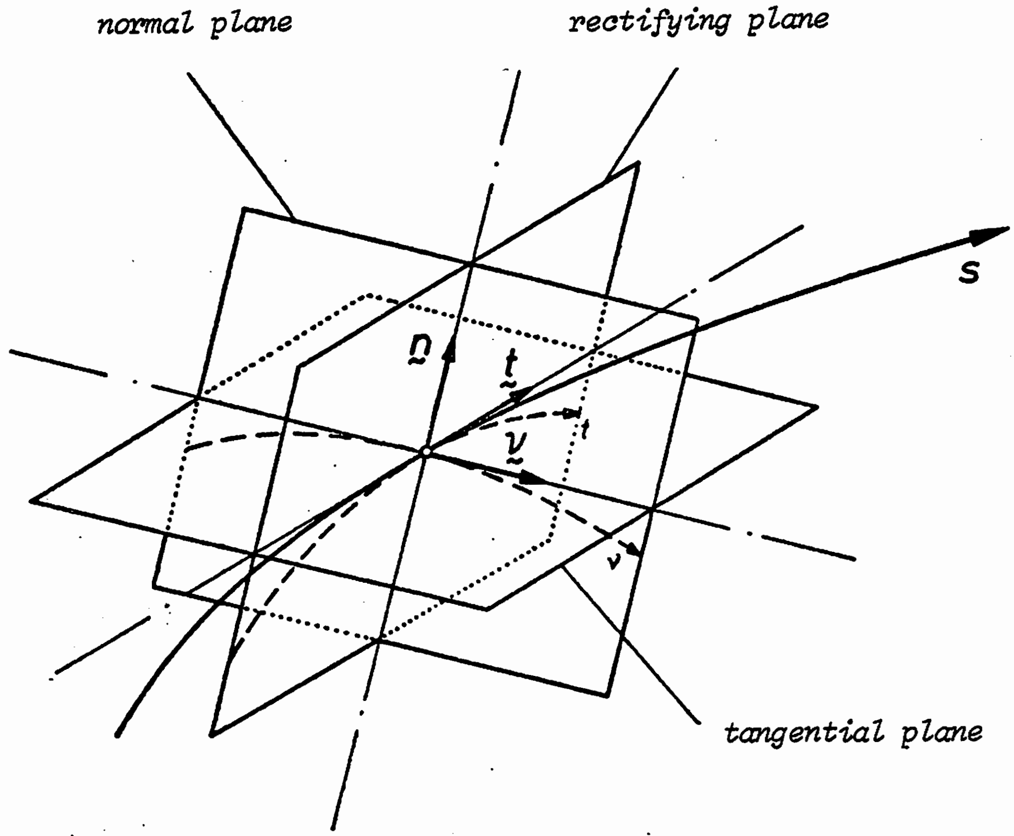

Fig. B1

Coordinate transformation of the system of the triad is performed by introducing

$$\left.\begin{array}{lll} \Theta'^{1} = t \;, & \underset{\sim}{a}'_{1} = \underset{\sim}{a}'^{1} = \underset{\sim}{t} \\ \Theta'^{2} = \nu \;, & \underset{\sim}{a}'_{2} = \underset{\sim}{a}'^{2} = \underset{\sim}{\nu} \end{array}\right\} \tag{B1.1}$$

into (A2.3) to (A2.5).

This generates the transformation formulae

$$\left.\begin{array}{l} \underset{\sim}{a}^{\kappa} = t^{\kappa}\underset{\sim}{t} + \nu^{\kappa}\underset{\sim}{\nu} \\ \underset{\sim}{a}_{\kappa} = t_{\kappa}\underset{\sim}{t} + \nu_{\kappa}\underset{\sim}{\nu} \end{array}\right\} \tag{B1.2}$$

where

$$t^{\alpha} = \frac{\partial \theta^{\alpha}}{\partial t} \quad , \quad t_{\alpha} = \frac{\partial t}{\partial \theta^{\alpha}}$$

$$\nu^{\alpha} = \frac{\partial \theta^{\alpha}}{\partial \nu} \quad , \quad \nu_{\alpha} = \frac{\partial \nu}{\partial \theta^{\alpha}} \; .$$

The transformations of (B1.2) allow to refer each tensor at a point $c \in C$ to the basis of the parameter lines t and ν. For instance, a vector $\underset{\sim}{u}$ is transformed as follows:

$$\underset{\sim}{u} = u_{\alpha}\, \underset{\sim}{a}^{\alpha} = u_{\alpha}\,(t^{\alpha}\, \underset{\sim}{t} + \nu^{\alpha}\, \underset{\sim}{\nu}) = u_t\, \underset{\sim}{t} + u_{\nu}\, \underset{\sim}{\nu} \qquad \text{(B1.3)}$$

with

$$\left. \begin{aligned} u_t &= u_{\alpha}\, t^{\alpha} \\ u_{\nu} &= u_{\alpha}\, \nu^{\alpha} \end{aligned} \right\} . \qquad \text{(B1.4)}$$

Proceeding from cartesian coordinates the Nabla-vector $\underset{\sim}{\nabla}$ is transformed to the surface basis by (A1.5) and then to the boundary basis by (B1.2).

$$\begin{aligned} \underset{\sim}{\nabla} &= \underset{\sim}{i}^{k} \frac{\partial}{\partial x^{k}} = \underset{\sim}{i}^{k} \frac{\partial \theta^{\alpha}}{\partial x^{k}} \frac{\partial}{\partial \theta^{\alpha}} = \underset{\sim}{a}^{\alpha} \frac{\partial}{\partial \theta^{\alpha}} \\ &= \underset{\sim}{t} \frac{\partial}{\partial t} + \underset{\sim}{\nu} \frac{\partial}{\partial \nu} \end{aligned}$$

A second order tensor $\underset{\sim}{T}$ undergoes a quite similar procedure as a vector.

$$\begin{aligned} \underset{\sim}{T} &= T_{\alpha\beta}\, \underset{\sim}{a}^{\alpha} \otimes \underset{\sim}{a}^{\beta} \\ &= T_{tt}\, \underset{\sim}{t} \otimes \underset{\sim}{t} + T_{t\nu}\, \underset{\sim}{t} \otimes \underset{\sim}{\nu} + T_{\nu t}\, \underset{\sim}{\nu} \otimes \underset{\sim}{t} + T_{\nu\nu}\, \underset{\sim}{\nu} \otimes \underset{\sim}{\nu} \end{aligned} \qquad \text{(B1.5)}$$

where

$$\left. \begin{aligned} T_{tt} &= T_{\alpha\beta}\, t^{\alpha} t^{\beta} \quad , \quad T_{\nu t} = T_{\alpha\beta}\, \nu^{\alpha} t^{\beta} \\ T_{t\nu} &= T_{\alpha\beta}\, t^{\alpha} \nu^{\beta} \quad , \quad T_{\nu\nu} = T_{\alpha\beta}\, \nu^{\alpha} \nu^{\beta} \end{aligned} \right\} \qquad \text{(B1.6)}$$

Chapter B2 refers to the transformation of a third ordered tensor.

$$\underset{\sim}{T} = T_{\alpha\beta\gamma}\, \underset{\sim}{a}^{\alpha} \otimes \underset{\sim}{a}^{\beta} \otimes \underset{\sim}{a}^{\gamma}$$

$$\begin{aligned} &= T_{ttt}\, \underset{\sim}{t} \otimes \underset{\sim}{t} \otimes \underset{\sim}{t} + T_{tt\nu}\, \underset{\sim}{t} \otimes \underset{\sim}{t} \otimes \underset{\sim}{\nu} + T_{t\nu t}\, \underset{\sim}{t} \otimes \underset{\sim}{\nu} \otimes \underset{\sim}{t} \\ &\quad + T_{t\nu\nu}\, \underset{\sim}{t} \otimes \nu \otimes \nu + T_{\nu tt}\, \underset{\sim}{\nu} \otimes \underset{\sim}{t} \otimes \underset{\sim}{t} + T_{\nu t\nu}\, \underset{\sim}{\nu} \otimes \underset{\sim}{t} \otimes \underset{\sim}{\nu} \\ &\quad + T_{\nu\nu t}\, \underset{\sim}{t} \otimes \underset{\sim}{\nu} \otimes \underset{\sim}{t} + T_{\nu\nu\nu}\, \underset{\sim}{\nu} \otimes \underset{\sim}{\nu} \otimes \underset{\sim}{\nu} \end{aligned} \tag{B1.7}$$

The physical components are defined by

$$\left.\begin{aligned} T_{ttt} &= T_{\alpha\beta\gamma}\, t^{\alpha} t^{\beta} t^{\gamma} , & T_{\nu tt} &= T_{\alpha\beta\gamma}\, \nu^{\alpha} t^{\beta} t^{\gamma} \\ T_{tt\nu} &= T_{\alpha\beta\gamma}\, t^{\alpha} t^{\beta} \nu^{\gamma} ; & T_{\nu t\nu} &= T_{\alpha\beta\gamma}\, \nu^{\alpha} t^{\beta} \nu^{\gamma} \\ T_{t\nu t} &= T_{\alpha\beta\gamma}\, t^{\alpha} \nu^{\beta} t^{\gamma} , & T_{\nu\nu t} &= T_{\alpha\beta\gamma}\, \nu^{\alpha} \nu^{\beta} t^{\gamma} \\ T_{t\nu\nu} &= T_{\alpha\beta\gamma}\, t^{\alpha} \nu^{\beta} \nu^{\gamma} , & T_{\nu\nu\nu} &= T_{\alpha\beta\gamma}\, \nu^{\alpha} \nu^{\beta} \nu^{\gamma} \end{aligned}\right\} . \tag{B1.8}$$

Otherwise, the components of the tensors are reobtained by scalar products.

$$\begin{aligned} u_{\alpha} &= \underset{\sim}{u} \cdot \underset{\sim}{a}_{\alpha} \\ &= u_t\, t_{\alpha} + u_{\nu}\, \nu_{\alpha} \end{aligned} \tag{B1.9}$$

$$\begin{aligned} T_{\alpha\beta} &= \underset{\sim}{T} \cdot\cdot\, \underset{\sim}{a}_{\beta} \otimes \underset{\sim}{a}_{\alpha} \\ &= T_{tt}\, t_{\alpha} t_{\beta} + T_{t\nu}\, t_{\alpha} \nu_{\beta} + T_{\nu t}\, \nu_{\alpha} t_{\beta} + T_{\nu\nu}\, \nu_{\alpha} \nu_{\beta} \end{aligned} \tag{B1.10}$$

$$\begin{aligned} T_{\alpha\beta\gamma} &= \underset{\sim}{T} \cdot\cdot\cdot\, \underset{\sim}{a}_{\beta} \otimes \underset{\sim}{a}_{\alpha} \otimes \underset{\sim}{a}_{\gamma} \\ &= T_{ttt}\, t_{\alpha} t_{\beta} t_{\gamma} + T_{tt\nu}\, t_{\alpha} t_{\beta} \nu_{\gamma} + T_{t\nu t}\, t_{\alpha} \nu_{\beta} t_{\gamma} \\ &\quad + T_{t\nu\nu}\, t_{\alpha} \nu_{\beta} \nu_{\gamma} + T_{\nu tt}\, \nu_{\alpha} t_{\beta} t_{\gamma} + T_{\nu t\nu}\, \nu_{\alpha} t_{\beta} \nu_{\gamma} \\ &\quad + T_{\nu\nu t}\, \nu_{\alpha} \nu_{\beta} t_{\gamma} + T_{\nu\nu\nu}\, \nu_{\alpha} \nu_{\beta} \nu_{\gamma} + \end{aligned} \tag{B1.11}$$

It is evident that raising up and down of indices does not change anything.

B2 Differentiation in the system of the triad

Within the infinite variety of possibilities, to determine normal and tangential derivatives of tensors, only derivatives with respect to the parameters s,t and ν are significant (s. figure B1).
First of all, the boundary base vectors $\underset{\sim}{t}$, $\underset{\sim}{\nu}$, $\underset{\sim}{n}$ shall be differentiated with respect to s by the formulae of Burali-Forti [37], [47].

$$\left.\begin{aligned} \underset{\sim}{\nu}_{,s} &= \kappa_t \underset{\sim}{t} + b_{\nu t} \underset{\sim}{n} \\ \underset{\sim}{t}_{,s} &= b_{tt} \underset{\sim}{n} - \kappa_\nu \underset{\sim}{\nu} \\ \underset{\sim}{n}_{,s} &= - b_{\nu t} \underset{\sim}{\nu} - b_{tt} \underset{\sim}{t} \end{aligned}\right\} \qquad \text{(B2.1)}$$

Since the *geodesic curvatures* κ_t and κ_ν of the parameter lines t and ν are equal to zero in the *tangential plane* the other derivatives follow immediately.

$$\left.\begin{aligned} \underset{\sim}{\nu}_{,t} &= b_{\nu t} \underset{\sim}{n} \\ \underset{\sim}{t}_{,t} &= b_{tt} \underset{\sim}{t} \\ \underset{\sim}{n}_{,t} &= - b_{\nu t} \underset{\sim}{\nu} - b_{tt} \underset{\sim}{t} \end{aligned}\right\} \qquad \text{(B2.2)}$$

$$\left.\begin{aligned} \underset{\sim}{\nu}_{,\nu} &= b_{\nu\nu} \underset{\sim}{n} \\ \underset{\sim}{t}_{,\nu} &= b_{\nu t} \underset{\sim}{n} \\ \underset{\sim}{n}_{,\nu} &= - b_{\nu t} \underset{\sim}{t} - b_{\nu\nu} \underset{\sim}{\nu} \end{aligned}\right\} \qquad \text{(B2.3)}$$

The quantities

$$b_{tt} = b_{\alpha\beta} t^\alpha t^\beta$$

$$b_{\nu\nu} = b_{\kappa\beta} \nu^\kappa \nu^\beta$$

stand for measures of *normal curvature* in the *rectifying* and the *normal plane*. The *torsion* of the surface is measured by

$$b_{\nu t} = b_{\kappa\beta} \nu^\kappa t^\beta .$$

Now, $\underset{\sim}{T}$ may be a tensor of order $\underset{\sim}{n}$. The total differential is given by

$$d\underset{\sim}{T} = d\underset{\sim}{r} \cdot \underset{\sim}{\nabla}\underset{\sim}{T} .$$

In consequence of

$$\frac{d\underset{\sim}{r}}{ds} = \frac{d\underset{\sim}{r}}{dt} = \underset{\sim}{t}$$

the following two directional derivatives

$$\frac{d\underset{\sim}{T}}{ds} = \frac{d\underset{\sim}{r}}{ds} \cdot \underset{\sim}{\nabla}\underset{\sim}{T}$$

$$\frac{d\underset{\sim}{T}}{dt} = \frac{d\underset{\sim}{r}}{dt} \cdot \underset{\sim}{\nabla}\underset{\sim}{T}$$

are identical. Since between different directional derivatives must be distinguished, the notation for partial differentiation is recommendable, such that

$$\frac{\partial\underset{\sim}{T}}{\partial s} = \frac{\partial\underset{\sim}{T}}{\partial t} . \qquad \text{(B2.4)}$$

If $\underset{\sim}{T}$ represents a scalar quantity ϕ, formula (B2.4) simply leads to

$$\phi_{,s} = \phi_{,t} . \qquad \text{(B2.5)}$$

Quite different facts exist already with regard to a vector, for instance a space vector.

$$\underset{\sim}{u} = u_\nu \underset{\sim}{\nu} + u_t \underset{\sim}{t} + u_3 \underset{\sim}{n}$$

Partial differentiation with respect to s and t yields

$$\underset{\sim}{u}_{,s} = u_{\nu,s} \underset{\sim}{\nu} + u_\nu \underset{\sim}{\nu}_{,s} + u_{t,s} \underset{\sim}{t} + u_t \underset{\sim}{t}_{,s} + u_{3,s} \underset{\sim}{n} + u_3 \underset{\sim}{n}_{,s}$$

$$\underset{\sim}{u}_{,t} = u_{\nu,t} \underset{\sim}{\nu} + u_\nu \underset{\sim}{\nu}_{,t} + u_{t,t} \underset{\sim}{t} + u_t \underset{\sim}{t}_{,t} + u_{3,t} \underset{\sim}{n} + u_3 \underset{\sim}{n}_{,t} .$$

From the equality of (B2.4) and by use of (B2.1) and (B2.2) the following derivatives are concluded hereafter:

$$\left.\begin{aligned} u_{\nu,t} &= u_{\nu,s} - \varkappa_t u_t \\ u_{t,t} &= u_{t,s} + \varkappa_t u_\nu \\ u_{3,t} &= u_{3,s} \end{aligned}\right\} \qquad \text{(B2.6)}$$

If the components of the gradient of a scalar ϕ

$$\underset{\sim}{\nabla}\phi = \phi_{,\nu}\,\underset{\sim}{\nu} + \phi_{,t}\,\underset{\sim}{t}$$

are introduced into the first two relations of (B2.6) derivatives of second order arise for ϕ.

$$\left.\begin{aligned} \phi_{,\nu t} &= \phi_{,\nu s} - \varkappa_t\,\phi_{,t} \\ \phi_{,tt} &= \phi_{,ss} + \varkappa_t\,\phi_{,\nu} \end{aligned}\right\} \qquad \text{(B2.7)}$$

Relation (B1.5) represents a second ordered tensor in the basis of the triad.

$$\underset{\sim}{T} = T_{tt}\,\underset{\sim}{t}\otimes\underset{\sim}{t} + T_{t\nu}\,\underset{\sim}{t}\otimes\underset{\sim}{\nu} + T_{\nu t}\,\underset{\sim}{\nu}\otimes\underset{\sim}{t} + T_{\nu\nu}\,\underset{\sim}{\nu}\otimes\underset{\sim}{\nu}$$

On ground of (B2.4) partial derivatives of the physical components have the following form:

$$\left.\begin{aligned} T_{tt,t} &= T_{tt,s} + \varkappa_t\,(T_{t\nu} + T_{\nu t}) \\ T_{t\nu,t} &= T_{t\nu,s} + \varkappa_t\,(T_{\nu\nu} - T_{tt}) \\ T_{\nu t,t} &= T_{\nu t,s} + \varkappa_t\,(T_{\nu\nu} - T_{tt}) \\ T_{\nu\nu,t} &= T_{\nu\nu,s} + \varkappa_t\,(T_{t\nu} + T_{\nu t}) \end{aligned}\right\} \qquad \text{(B2.8)}$$

It is also possible to transform covariant derivatives to the basis of the triad. This may be demonstrated by means of a second order tensor $\underset{\sim}{T}$, where use is made of (A3.8) and (B1.2).

$$\begin{aligned} \underset{\sim}{T}_{,s} &= (T_{\varkappa\lambda}\,\underset{\sim}{a}^{\varkappa}\otimes\underset{\sim}{a}^{\lambda})_{,\gamma}\,t^{\gamma} \\ &= T_{\varkappa\lambda|\gamma}\,t^{\varkappa}t^{\lambda}t^{\gamma}\,\underset{\sim}{t}\otimes\underset{\sim}{t} + T_{\varkappa\lambda|\gamma}\,t^{\varkappa}\nu^{\lambda}t^{\gamma}\,\underset{\sim}{t}\otimes\underset{\sim}{\nu} \\ &\quad + T_{\varkappa\lambda|\gamma}\,\nu^{\varkappa}t^{\lambda}t^{\gamma}\,\underset{\sim}{\nu}\otimes\underset{\sim}{t} + T_{\varkappa\lambda|\gamma}\,\nu^{\varkappa}\nu^{\lambda}t^{\gamma}\,\underset{\sim}{\nu}\otimes\underset{\sim}{\nu} \\ &\quad + (b_{tt}T_{tt} + b_{\nu t}T_{\nu t})\,\underset{\sim}{n}\otimes\underset{\sim}{t} + (b_{tt}T_{t\nu} + b_{\nu t}T_{\nu\nu})\,\underset{\sim}{n}\otimes\underset{\sim}{\nu} \\ &\quad + (b_{tt}T_{tt} + b_{t\nu}T_{t\nu})\,\underset{\sim}{t}\otimes\underset{\sim}{n} + (b_{tt}T_{\nu t} + b_{t\nu}T_{\nu\nu})\,\underset{\sim}{\nu}\otimes\underset{\sim}{n} \end{aligned} \qquad \text{(B2.9)}$$

On the other hand, partial differentiation of (B1.5) with respect to t yields by paying attention to (B2.2):

$$\begin{aligned}
&\underset{\sim}{T}_{,\mu} \\
&= T_{\mu\mu,\mu}\, \underset{\sim}{t} \otimes \underset{\sim}{t} + T_{\mu\nu,\mu}\, \underset{\sim}{t} \otimes \underset{\sim}{v} + T_{\nu\mu,\mu}\, \underset{\sim}{v} \otimes \underset{\sim}{t} + T_{\nu\nu,\mu}\, \underset{\sim}{v} \otimes \underset{\sim}{v} \\
&\quad + (b_{\mu\mu} T_{\mu\mu} + b_{\nu\mu} T_{\nu\mu})\, \underset{\sim}{n} \otimes \underset{\sim}{t} + (b_{\mu\mu} T_{\mu\nu} + b_{\nu\mu} T_{\nu\nu})\, \underset{\sim}{n} \otimes \underset{\sim}{v} \\
&\quad + (b_{\mu\mu} T_{\mu\mu} + b_{\mu\nu} T_{\mu\nu})\, \underset{\sim}{t} \otimes \underset{\sim}{n} + (b_{\mu\mu} T_{\nu\mu} + b_{\mu\nu} T_{\nu\nu})\, \underset{\sim}{v} \otimes \underset{\sim}{n}
\end{aligned} \qquad \text{(B2.10)}$$

As you see, simple relations originate from the equality of (B2.9) and (B2.10) due to (B2.4).

$$\left.\begin{aligned}
T_{\mu\mu,\mu} &= T_{\alpha\beta|\gamma}\, t^\alpha t^\beta t^\gamma \\
T_{\mu\nu,\mu} &= T_{\alpha\beta|\gamma}\, t^\alpha \nu^\beta t^\gamma \\
T_{\nu\mu,\mu} &= T_{\alpha\beta|\gamma}\, \nu^\alpha t^\beta t^\gamma \\
T_{\nu\nu,\mu} &= T_{\alpha\beta|\gamma}\, \nu^\alpha \nu^\beta t^\gamma
\end{aligned}\right\} \qquad \text{(B2.11)}$$

In the same way, derivatives referred to the parameter line ν are developed.

$$\left.\begin{aligned}
T_{\mu\mu,\nu} &= T_{\alpha\beta|\gamma}\, t^\alpha t^\beta \nu^\gamma \\
T_{\mu\nu,\nu} &= T_{\alpha\beta|\gamma}\, t^\alpha \nu^\beta \nu^\gamma \\
T_{\nu\mu,\nu} &= T_{\alpha\beta|\gamma}\, \nu^\alpha t^\beta \nu^\gamma \\
T_{\nu\nu,\nu} &= T_{\alpha\beta|\gamma}\, \nu^\alpha \nu^\beta \nu^\gamma
\end{aligned}\right\} \qquad \text{(B2.12)}$$

The componenets $T_{\alpha\beta|\lambda}$ belong to a tensor of third order. In accordance with (B1.7) and (B1.8) they are recovered by (B1.11) if relations (B2.11) and (B2.12) are taken into account additionally.

$$\begin{aligned}
T_{\alpha\beta|\gamma} = {}& T_{\mu\mu,\mu}\, t_\alpha t_\beta t_\gamma + T_{\mu\mu,\nu}\, t_\alpha t_\beta \nu_\gamma + T_{\mu\nu,\mu}\, t_\alpha \nu_\beta t_\gamma \\
& + T_{\mu\nu,\nu}\, t_\alpha \nu_\beta \nu_\gamma + T_{\nu\mu,\mu}\, \nu_\alpha t_\beta \nu_\gamma + T_{\nu\mu,\nu}\, \nu_\alpha t_\beta \nu_\gamma \\
& + T_{\nu\nu,\mu}\, \nu_\alpha \nu_\beta t_\gamma + T_{\nu\nu,\nu}\, \nu_\alpha \nu_\beta \nu_\gamma
\end{aligned} \qquad \text{(B2.13)}$$

Finally, contractions of $T^{\alpha\beta}|_\lambda$ enforce very simple transformations, which are needed below.

$$T^{\alpha\beta}|_\alpha = (T_{tt,t} + T_{\nu t,\nu})\, t^\beta + (T_{t\nu,t} + T_{\nu\nu,\nu})\, \nu^\beta \qquad \text{(B2.14)}$$

$$T^{\alpha\beta}|_\beta = (T_{tt,t} + T_{t\nu,\nu})\, t^\alpha + (T_{\nu t,t} + T_{\nu\nu,\nu})\, \nu^\alpha \qquad \text{(B2.15)}$$

APPENDIX C

C1 Generalized integral theorems

Let there be any given product o between the Nabla-operator $\underset{\sim}{\nabla}$ and any scalar, vectorial or tensorial field function $\underset{\sim}{\Phi}(\underset{\sim}{r})$. Then Gauß theorem in tensor analysis of threedimensional space can be formulated completely general [39], [40], [42]. The product referred to can be given as scalar or vector product or as product of dyades. Now, the *generalized integral theorem of Gauß* is formulated by

$$\int_V \underset{\sim}{\nabla} \circ \underset{\sim}{\Phi}\, dV = \int_F \underset{\sim}{n} \circ \underset{\sim}{\Phi}\, dF \tag{C1.1}$$

or

$$\int_V \underset{\sim}{\Phi} \circ \underset{\sim}{\overleftarrow{\nabla}}\, dV = \int_F \underset{\sim}{\Phi} \circ \underset{\sim}{n}\, dF . \tag{C1.2}$$

The meaning of $\underset{\sim}{n}$ is obvious; it is the unit normal vector of a differential area dF. The Nabla-vector $\underset{\sim}{\overleftarrow{\nabla}}$ (Nabla-left) differentiates the factors standing on the left hand side.

If two field functions $\underset{\sim}{\Phi}(\underset{\sim}{r})$ and $\underset{\sim}{\Psi}(\underset{\sim}{r})$ are related to the Nabla-vector by different products, then it follows for example:

$$\int_V (\underset{\sim}{\nabla} \circ \underset{\sim}{\Phi}) \circ \underset{\sim}{\Psi}\, dV = \int_F (\underset{\sim}{n} \circ \underset{\sim}{\Phi}) \circ \underset{\sim}{\Psi}\, dF \tag{C1.3}$$

$$\int_V \overset{\downarrow}{\underset{\sim}{\Phi}} \circ \underset{\sim}{\nabla} \circ \overset{\downarrow}{\underset{\sim}{\Psi}}\, dV = \int_F \underset{\sim}{\Phi} \circ \underset{\sim}{n} \circ \underset{\sim}{\Psi}\, dF \tag{C1.4}$$

A very simple instruction is offered by Lagally [40] in order to perform integration by parts, even if the differential expression in question is rather complicated. This shall be clarified with reference to example (C1.4). After carrying out the operation required in the volume integral, it follows in detail

$$\int_V \overset{\downarrow}{\underset{\sim}{\Phi}} \circ \underset{\sim}{\nabla} \circ \overset{\downarrow}{\underset{\sim}{\Psi}}\, dV$$

$$= \int_V \underset{\sim}{\Phi} \circ \underset{\sim}{\nabla} \circ \underset{\sim}{\Psi}\, dV + \int_V \underset{\sim}{\Phi} \circ \underset{\sim}{\overleftarrow{\nabla}} \circ \underset{\sim}{\Psi}\, dV = \int_F \underset{\sim}{\Phi} \circ \underset{\sim}{n} \circ \underset{\sim}{\Psi}\, dF$$

or

$$\int_V \underset{\sim}{\Phi} \circ \underset{\sim}{\nabla} \circ \underset{\sim}{\Psi}\, dV = \int_F \underset{\sim}{\Phi} \circ \underset{\sim}{n} \circ \underset{\sim}{\Psi}\, dF - \int_V \underset{\sim}{\Phi} \circ \overset{\leftarrow}{\underset{\sim}{\nabla}} \circ \underset{\sim}{\Psi}\, dV. \qquad \text{(C1.5)}$$

By means of (C1.5) the procedure for partial integration of a volume integral is clear: After integration by parts the unit normal appears in the surface integral instead of the Nabla-vector. The next step is to subtract a further volume integral from the surface integral. This volume integral is of the same form as the original one, but with the difference that Nabla-left has to be inserted.

In Euclidean space the Nabla-operator can be represented in a half-symbolic description, so called by Klingbeil [48].

$$\underset{\sim}{\nabla} = \underset{\sim}{g}^{i}\, \nabla_i = \underset{\sim}{g}^{i} (\dots)|_i \qquad \text{(C1.6)}$$

In this form the components ∇_i generate immediately covariant derivatives of tensor components. The basis of the tensor however remains unaffected.

The generalized integral theorem of Gauß permits a generalization of Green's integral formulae. In order to derive the *second formula of Green*, integration by parts is applied to a proper integral and that is

$$\int_V \underset{\sim}{\Phi} \circ \underset{\sim}{\nabla} \circ \underset{\sim}{\nabla} \circ \underset{\sim}{\Psi}\, dV$$

$$= \int_F \underset{\sim}{\Phi} \circ \underset{\sim}{n} \circ \underset{\sim}{\nabla} \circ \underset{\sim}{\Psi}\, dF - \int_V \underset{\sim}{\Phi} \circ \overset{\leftarrow}{\underset{\sim}{\nabla}} \circ \underset{\sim}{\nabla} \circ \underset{\sim}{\Psi}\, dV.$$

The second integral on the right is partially integrated once more and this leads at last to the result

$$\int_V [\underset{\sim}{\Phi} \circ \overset{\rightarrow}{\underset{\sim}{\nabla}} \circ \overset{\rightarrow}{\underset{\sim}{\nabla}} \circ \underset{\sim}{\Psi} - \underset{\sim}{\Phi} \circ \overset{\leftarrow}{\underset{\sim}{\nabla}} \circ \overset{\leftarrow}{\underset{\sim}{\nabla}} \circ \underset{\sim}{\Psi}]\, dV$$

$$= \int_F [\underset{\sim}{\Phi} \circ \underset{\sim}{n} \circ \overset{\rightarrow}{\underset{\sim}{\nabla}} \circ \underset{\sim}{\Psi} - \underset{\sim}{\Phi} \circ \overset{\leftarrow}{\underset{\sim}{\nabla}} \circ \underset{\sim}{n} \circ \underset{\sim}{\Psi}]\, dF. \qquad \text{(C1.7)}$$

For the sake of clearness, all Nabla-right operators are also shown off by sign in analogy to Nabla-left.

In special case of scalar functions for Φ and Ψ the second formula of Green appears in that form, which is well-known in the analysis of real functions.

$$\int_V (\phi\,\psi|_i^{\,i} - \psi\,\phi|_i^{\,i})\,dV = \int_F (\phi\,\psi_{,n} - \psi\,\phi_{,n})\,dF$$

C2 Special cases of generalized integral theorems on plane surfaces

Be given ψ_α, ψ_3 as components of a space vector $\underset{\sim}{\Psi}$. Moreover, ϕ may be a scalar function and $T^{\alpha\beta}$ components of the surface tensor $\underset{\approx}{T}$ of second order. Integration by parts shall be applied to the plane surface integrals

$$\int_A T^{\alpha\beta}\,\psi_\alpha|_\beta\,dA \tag{C2.1}$$

$$\int_A T^{\alpha\beta}\,\psi_3|_{\alpha\beta}\,dA \tag{C2.2}$$

$$\int_A T_{\alpha\beta}\,\epsilon^{\alpha\rho}\epsilon^{\beta\lambda}\,\phi|_{\rho\lambda}\,dA. \tag{C2.3}$$

Integration of (C2.1) yields with appropriately chosen products by (C1.5) in the plane

$$\int_A \underset{\sim}{T}\cdot\cdot\,\underset{\sim}{\nabla}\,\underset{\sim}{\Psi}\,dA$$

$$= \oint_C \underset{\sim}{T}\cdot\cdot\,\underset{\sim}{\nu}\,\underset{\sim}{\Psi}\,ds - \int_A \underset{\sim}{T}\cdot\cdot\,\overleftarrow{\underset{\sim}{\nabla}}\,\underset{\sim}{\Psi}\,dA.$$

The change from symbolic to tensorial description is easily performed. With (C1.6) in the plane it follows in detail for the integrands:

$$\underset{\sim}{T}\cdot\cdot\,\underset{\sim}{\nabla}\,\underset{\sim}{\Psi} = T^{\alpha\beta}\,\underset{\sim}{a}_\alpha\otimes\underset{\sim}{a}_\beta\cdot\cdot\,\nabla_\rho\,\underset{\sim}{a}^\rho(\psi_\lambda\,\underset{\sim}{a}^\lambda + \psi_3\,\underset{\sim}{n})$$

$$= T^{\alpha\beta}\,\psi_\alpha|_\beta$$

$$\underset{\sim}{T} \cdot\cdot \overset{\leftarrow}{\underset{\sim}{\nabla}} \underset{\sim}{\Psi} = T^{\alpha\beta} \underset{\sim}{a}_\alpha \otimes \underset{\sim}{a}_\beta \cdot\cdot \overset{\leftarrow}{\nabla}_\rho \underset{\sim}{a}^\rho (\Psi_\lambda \underset{\sim}{a}^\lambda + \Psi_3 \underset{\sim}{n})$$
$$= T^{\alpha\beta}|_\beta \Psi_\alpha$$

$$\underset{\sim}{T} \cdot\cdot \underset{\sim}{\nu} \underset{\sim}{\Psi} = T^{\alpha\beta} \underset{\sim}{a}_\alpha \otimes \underset{\sim}{a}_\beta \cdot\cdot \nu_\rho \underset{\sim}{a}^\rho (\Psi_\lambda \underset{\sim}{a}^\lambda + \Psi_3 \underset{\sim}{n})$$
$$= T_{\nu\nu} \Psi_\nu + T_{t\nu} \Psi_t$$

Finally it follows

$$\int_A T^{\alpha\beta} \Psi_\alpha|_\beta \, dA$$
$$= -\int_A T^{\alpha\beta}|_\beta \Psi_\alpha \, dA + \oint_C (T_{\nu\nu} \Psi_\nu + T_{t\nu} \Psi_t) \, ds . \qquad \text{(C2.4)}$$

For integration of (C2.2) Green's second formula (C1.7) is available. The third component of $\underset{\sim}{\Psi}$ emerges from $\psi_3 = \underset{\sim}{\psi} \cdot \underset{\sim}{n}$.

$$\int_A [\underset{\sim}{T} \cdot\cdot \overset{\rightarrow}{\underset{\sim}{\nabla}} \overset{\rightarrow}{\underset{\sim}{\nabla}} \Psi_3 - \underset{\sim}{T} \cdot\cdot \overset{\leftarrow}{\underset{\sim}{\nabla}} \overset{\leftarrow}{\underset{\sim}{\nabla}} \Psi_3] \, dA$$
$$= \oint_C [\underset{\sim}{T} \cdot\cdot \underset{\sim}{\nu} \overset{\rightarrow}{\underset{\sim}{\nabla}} \Psi_3 - \underset{\sim}{T} \cdot\cdot \overset{\leftarrow}{\underset{\sim}{\nabla}} \underset{\sim}{\nu} \Psi_3] \, ds \qquad \text{(C2.5)}$$

If only Nabla-right operators $\underset{\sim}{\nabla}$ are chosen, this leads after transposition to

$$\int_A [\underset{\sim}{T} \cdot\cdot \underset{\sim}{\nabla} \underset{\sim}{\nabla} \Psi_3 - \Psi_3 \underset{\sim}{\nabla} \underset{\sim}{\nabla} \cdot\cdot \underset{\sim}{T}^T] \, dA$$
$$\oint_C [\underset{\sim}{T} \cdot\cdot \underset{\sim}{\nu} \underset{\sim}{\nabla} \Psi_3 - \Psi_3 \underset{\sim}{\nu} \underset{\sim}{\nabla} \cdot\cdot \underset{\sim}{T}^T] \, ds .$$

In special remind of (A3.6), (B1.10) and (B2.15) the integrands represent as follows:

$$\underset{\sim}{T} \cdot\cdot \underset{\sim}{\nabla}\,\underset{\sim}{\nabla}\, \Psi_3 = T^{\alpha\beta}\, \Psi_3 /_{\alpha\beta}$$
$$\Psi_3\, \underset{\sim}{\nabla}\,\underset{\sim}{\nabla} \cdot\cdot \underset{\sim}{T}^T = \Psi_3\, T^{\alpha\beta}/_{\beta\alpha}$$
$$\underset{\sim}{T} \cdot\cdot \underset{\sim}{\nu}\,\underset{\sim}{\nabla}\, \Psi_3 = T_{\nu\nu}\, \Psi_{3,\nu} + T_{t\nu}\, \Psi_{3,t}$$
$$\Psi_3\, \underset{\sim}{\nu}\,\underset{\sim}{\nabla} \cdot\cdot \underset{\sim}{T}^T = (T_{\nu t,t} + T_{\nu\nu,\nu})\, \Psi_3$$

The last four relations generate a description in components for the transformation of (C2.5).

$$\int_A [\, T^{\alpha\beta}\, \Psi_3 /_{\alpha\beta} - T^{\alpha\beta}/_{\beta\alpha}\, \Psi_3 \,]\, dA$$
$$= \oint_C [\, T_{\nu\nu}\, \Psi_{3,\nu} + T_{t\nu}\, \Psi_{3,t} - (T_{\nu t,t} + T_{\nu\nu,\nu})\, \Psi_3 \,]\, ds \tag{C2.6}$$

Because of $\psi_{3,t} = \psi_{3,s}$ in view of (B2.6), partial integration can be applied once more and this provides the final form of the transformation (C2.5).

$$\int_A [\, T^{\alpha\beta}\, \Psi_3 /_{\alpha\beta} - T^{\alpha\beta}/_{\beta\alpha}\, \Psi_3 \,]\, dA$$
$$= \oint_C [\, T_{\nu\nu}\, \Psi_{3,\nu} - (T_{t\nu,t} + T_{\nu\nu,\nu} + T_{t\nu,s})\, \Psi_3 \,]\, ds - [\, T_{t\nu}\, \Psi_3 \,]_C \tag{C2.7}$$

The expression

$$[\, T_{t\nu}\, \Psi_3 \,]_C = \sum_{i=1}^{N} [\, T_{t\nu}(s_i + 0) - T_{t\nu}(s_i - 0) \,]\, \Psi_3(s_i)$$

has to be taken into account at all corners s_i of the boundary curve C.

Partial integration of (C2.3) follows by the differential operator

$$\underset{\sim}{n} \times \underset{\sim}{\nabla} = \underset{\sim}{n} \times \underset{\sim}{I} \cdot \underset{\sim}{\nabla} = - \underset{\sim}{\epsilon} \cdot \underset{\sim}{\nabla} = \underset{\sim}{\nabla} \cdot \underset{\sim}{\epsilon}$$

also by using Green's second integral formula in the shape of (C1.7)

$$\int_A [\, \underset{\sim}{T} \cdot\cdot (\vec{\underset{\sim}{\nabla}} \cdot \underset{\sim}{\epsilon})(\vec{\underset{\sim}{\nabla}} \cdot \underset{\sim}{\epsilon})\, \phi - \underset{\sim}{T} \cdot\cdot (\overleftarrow{\underset{\sim}{\nabla}} \cdot \underset{\sim}{\epsilon})(\overleftarrow{\underset{\sim}{\nabla}} \cdot \underset{\sim}{\epsilon})\, \phi \,]\, dA$$
$$= \oint_C [\, \underset{\sim}{T} \cdot\cdot (\underset{\sim}{\nu} \cdot \underset{\sim}{\epsilon})(\vec{\underset{\sim}{\nabla}} \cdot \underset{\sim}{\epsilon})\, \phi - \underset{\sim}{T} \cdot\cdot (\overleftarrow{\underset{\sim}{\nabla}} \cdot \underset{\sim}{\epsilon})(\underset{\sim}{\nu} \cdot \underset{\sim}{\epsilon})\, \phi \,]\, ds$$

Since $\underset{\sim}{\epsilon}^T = -\underset{\sim}{\epsilon}$ and $\underset{\sim}{\nu} \cdot \underset{\sim}{\epsilon} = \underset{\sim}{t}$, it results after transposition

$$\int_A [\underset{\sim}{T} \cdot\cdot (\underset{\sim}{\nabla} \cdot \underset{\sim}{\epsilon})(\underset{\sim}{\nabla} \cdot \underset{\sim}{\epsilon}) \phi - \phi (\underset{\sim}{\nabla} \cdot \underset{\sim}{\epsilon})(\underset{\sim}{\nabla} \cdot \underset{\sim}{\epsilon}) \cdot\cdot \underset{\sim}{T}^T] \, dA$$

$$= \oint_C [\underset{\sim}{T} \cdot\cdot \underset{\sim}{t} (\underset{\sim}{\nabla} \cdot \underset{\sim}{\epsilon}) \phi - \phi \underset{\sim}{t} (\underset{\sim}{\nabla} \cdot \underset{\sim}{\epsilon}) \cdot\cdot \underset{\sim}{T}^T] \, ds . \tag{C2.8}$$

Successively, (C2.8) will be transformed to a tensor description, for which especially formulae (A1.12) and (B2.13) are used.

$$\begin{aligned}
\underset{\sim}{T} \cdot\cdot (\underset{\sim}{\nabla} \cdot \underset{\sim}{\epsilon})(\underset{\sim}{\nabla} \cdot \underset{\sim}{\epsilon}) \phi &= T_{\alpha\beta} \, \epsilon^{\alpha\rho} \epsilon^{\beta\lambda} \phi|_{\rho\lambda} \\
\phi (\underset{\sim}{\nabla} \cdot \underset{\sim}{\epsilon})(\underset{\sim}{\nabla} \cdot \underset{\sim}{\epsilon}) \cdot\cdot \underset{\sim}{T}^T &= \epsilon^{\alpha\rho} \epsilon^{\beta\lambda} T_{\beta\alpha}|_{\rho\lambda} \, \phi \\
\underset{\sim}{T} \cdot\cdot \underset{\sim}{t} (\underset{\sim}{\nabla} \cdot \underset{\sim}{\epsilon}) \phi &= T_{tt} \, \phi_{,\nu} - T_{\nu t} \, \phi_{,t} \\
\phi \underset{\sim}{t} (\underset{\sim}{\nabla} \cdot \underset{\sim}{\epsilon}) \cdot\cdot \underset{\sim}{T}^T &= (T_{tt,\nu} - T_{t\nu,t}) \, \phi
\end{aligned}$$

Introducing these relations into (C2.8) the tensorial formulation is obtained.

$$\int_A [\epsilon^{\alpha\rho} \epsilon^{\beta\lambda} T_{\alpha\beta} \, \phi|_{\rho\lambda} - \epsilon^{\alpha\rho} \epsilon^{\beta\lambda} T_{\beta\alpha}|_{\rho\lambda} \, \phi] \, dA$$

$$\oint_C [T_{tt} \, \phi_{,\nu} - T_{\nu t} \, \phi_{,t} - (T_{tt,\nu} - T_{t\nu,t}) \, \phi] \, ds \tag{C2.9}$$

Since $\phi_{,t} = \phi_{,3}$ with respect ot (B2.5), on the boundary curve C partial integration is performable again.

$$\int_A [\epsilon^{\alpha\rho} \epsilon^{\beta\lambda} T_{\alpha\beta} \, \phi|_{\rho\lambda} - \epsilon^{\alpha\rho} \epsilon^{\beta\lambda} T_{\beta\alpha}|_{\rho\lambda} \, \phi] \, dA$$

$$= \oint_C [T_{tt} \, \phi_{,\nu} - (T_{tt,\nu} - T_{t\nu,t} - T_{\nu t,s}) \, \phi] \, ds - [T_{\nu t} \, \phi]_C \tag{C2.10}$$

Similar to (C2.7), the singularities

$$[T_{\nu t} \, \phi]_C = \sum_{j=1}^{N} [T_{\nu t}(s_j + 0) - T_{\nu t}(s_j - 0)] \, \phi(s_j)$$

must be taken into consideration at corners s_i of the boundary curve C.

At the end of this chapter it shall be remarked, that form (C2.9) the *compatibility condition* is disclosed, if the scalar function is supposed to be $\phi \equiv 1$ everywhere and if the tensor $\underset{\sim}{T}$ is identified with the linear strain tensor $\underset{\sim}{\Theta}$ of (2.5) or (6.10). Then it follows from (C2.9)

$$\oint_C (\Theta_{tt,\nu} - \Theta_{t\nu,t})\, ds = \int_A \epsilon^{\kappa\rho}\epsilon^{\beta\lambda}\Theta_{\kappa\beta}|_{\rho\lambda}\, dA. \qquad \text{(C2.11)}$$

Herein, C may be any given closed curve and A may be the area enclosed by C. Because of

$$\Theta_{tt,\nu} - \Theta_{t\nu,t} = \varphi_{,s}$$

according to (5.3.8), the line integral of (C2.11) is equal to zero. This means, that the integrand of the surface integral must vanish identically.

$$\epsilon^{\alpha\rho}\epsilon^{\beta\lambda}\Theta_{\kappa\beta}|_{\rho\lambda} \equiv 0 \qquad \text{(C2.12)}$$

By (4.3.6) an alternative form for (C2.12) is permissible, and that is

$$\epsilon^{\beta\lambda}\varkappa_{\beta 3}|_{\lambda} \equiv 0. \qquad \text{(C2.13)}$$

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Personal record

14. 2. 1943	Born in Gdańsk as son of the businessman Erich Kolberg and his wife Elfriede, nêe Wiechert
1949 - 1954	Gymnasium in Cuxhaven
1954 - 1958	Gymnasium in Bad-Neuenahr
1958 - 1964	Gymnasium in Cuxhaven
18. 2. 1954	Leaving-examination
1964 - 1956	Military service
1966 - 1972	Study of mechanical engineering at the Technische Universität Berlin
15. 5. 1972	Diploma
1.6.1972 - 30.6.1973	Occupation at the department "Dynamik und Regeltechnik" of the group of research NEUE TECHNOLOGIE of the MAN-concern in Munich
Since 1. 7. 1973	Scientific assistent at the Lehrstuhl für Mechanik II des Instituts für Mechanik der Ruhr-Universität Bochum, o. Professor Dr.-Ing. H. Stumpf

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