RUHR-UNIVERSITÄT BOCHUM

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An Alternative Approach to the Elastic-Viscoplastic Initial-Boundary Value Problem

Heft Nr. 28

Mitteilungen
aus dem
Institut für Mechanik

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MITTEILUNGEN AUS DEM INSTITUT FÜR MECHANIK NR. 28
SEPTEMBER 1981

Editor:
Institut für Mechanik der Ruhr-Universität Bochum

Zusammenfassung

In dieser Arbeit wird eine vereinheitlichte mathematische Herleitung des dem Anfangsrandwertproblem für elasto-visko-plastische Körper bei großen Deformationen entsprechenden Minimum-Prinzips vorgestellt. Eine integrale Formulierung des Problems wird vorgeschlagen. Das Materialverhalten wird mittels interner Parameter beschrieben. Diskontinuitäten der Verschiebungen über interne Oberflächen sind zugelassen. Eine Konstruktion der dualen Räume der Dehnungs- und Spannungsfunktionen wird angegeben. Das resultierende Minimum-Prinzip, das der gesamten Geschichte des Deformationsprozesses entspricht, wird diskutiert.

Summary

The paper presents a unified mathematical derivation of the minimum principle corresponding to the initial-boundary value problem of large deformation of elastic-viscoplastic solids. An integral formulation of the problem is proposed. The material behaviour is described in terms of the internal parameters. Discontinuities in the displacements across internal surfaces are admitted. A construction of the dual spaces of strain and stress functions is given. The resulting minimum principle, corresponding to the entire history of the deformation process, is discussed.

ACKNOWLEDGMENTS

The basis for this article was laid during the period October to December 1980, when the author was a Visiting Professor in the Ruhr-Universität Bochum. The general idea of the work was presented at the seminars given by the author in the Institut für Mechanik II.

The author is indebted to Prof.Dr.H.Stumpf for his kind invitation to the Institut für Mechanik II, for stimulating discussion during the seminars and for his continuous interest and support.

The author also wishes to thank Dr.D.Weichert, Dr.F.Labisch and other participants of the seminars for a number of helpful discussions and comments concerning the work.

P.Rafalski

Warszawa, August 1981

CONTENTS

1. INTRODUCTION

This work presents a unified mathematical derivation of the minimum principle corresponding to the initial-boundary value problem of large deformations of elastic-viscoplastic solids. Although the presented description of the deformation and the material behaviour is based on the classical concepts of the theory of plasticity, it represents a considerable departure from the usual approach used in the literature. The essential features of the proposed description are:

- The definitions of the strain and stress tensors are based on the integral relations for the regular subregions of the considered region. Similarly the equilibrium equations for the continuous body take the primary form of the requirement of equilibrium for every regular subregion of the body.
- The concept of the generalized standard material proposed by NGUYEN and HALPHEN [3],[4] is used to describe the material behaviour. A modified form of this concept, where the internal and external parameters are related by the appropriate balance equations, is proposed. According to this modification the stress tensor represents the external loading of the material while the internal stress tensor represents the state of forces in the material structure. Similarly the strain tensor represents the external energy supplied to the material while the total internal strain tensor represents the energy absorbed in the structure. It should be noted that the internal stress and strain tensors are defined in the system of coordinates connected with the material structure.
- In contrast to the geometrical concept of the plastic deformation proposed by GREEN and NAGHDI [1] or LEE and GERMAIN [2] and commonly used in large deformations theory the plastic and elastic components of the internal strain are defined here as the representatives of the mechanical energy dissipated and stored in the material structure, respectively. Due to this definition the total internal

strain tensor is additively decomposed into the elastic and plastic part.

- In contrast to the restricted variational principles for the "rate problem" existing in the literature [11] we construct here the minimum principle for the entire history of the elastic-viscoplastic process, represented by the stress and strains functions defined in the space-time region. Similar construction of the minimum principle for small deformations of elastic-viscoplastic solids was presented in [14],[15]. The functional to be minimized is expressed in terms of the integrals over the considered space-time region. Due to the properties of the prescribed weight function of time, appearing in the integrands, the plastic strain function is integrable provided that the rate of this function is integrable. It should be emphasized that the rate of plastic strain is here a primary notion. The plastic strain function is uniquely determined by its rate and the initial value.
- In contrast to the dual spaces of tensor functions used in the mathematical theory of plasticity (such as the space of bounded deformations [12],[13]) the dual spaces constructed in the work are determined by the constitutive relations. This original construction, which ensures the reflexivity of the dual spaces, was already announced in [16],[17]. It should be noted that the reflexivity of the dual spaces is of fundamental importance in a proof of the existence of the solution of the problem.

The formal mathematical description of the problem and the construction of the corresponding minimum principle is based on the theory of the measure and the integral [7],[8], theory of the duality [6],[7],[9],[10] and the convex analysis [5],[6],[9],[10]. To provide a self-contained presentation of the work the main mathematical ideas and principles used in the derivations are exposed in the text.

In section 2 we introduce the concept of the regular region. The set of all regular regions of positive volume is used in the work to describe the continuous compatible deformation of the body. In order to generalize the considerations onto discontinuous compatible deformations we also introduce the additional set of all regular regions of volume zero. Such regions are identified with the slip surfaces.

The idea of continuous and discontinuous displacement function is presented in section 3. A particular construction of the continuous compatible displacement function is given in section 4. The concept of subdivision of the considered body into the regular regions subjected to the homogeneous deformation is similar to the finite element discretization and can be directly used to obtain the approximated solution.

In section 5 attention is confined to a regular region subjected to the homogeneous deformation. For such region the concept of the strain and stress, which will be used in further consideration, is defined.

Section 6 presents an attempt to provide a comprehensive material model based on the concept of the internal parameters. Extending the idea of the "generalized stress" and the "generalized strain", presented by NGUYEN in [3], to large deformations we introduce the "internal stress tensor" which determines the internal forces in the hypothetical material structure. Consequently the "elastic strain tensor" and the "plastic strain tensor" are interpreted as the internal deformations of the elastic and plastic components of the material structure.

The constitutive relations for the elastic-viscoplastic material described in section 7 are based on the theory of the "generalized standard materials" where one postulates the existence of convex free energy function and convex dissipation potential. The relations are postulated in the unified form of multi-valued functions and expressed with the standard notions of the convex analysis.

It is shown in section 8 that the descriptions of the plastic flow named after HUBER-VON MISES or TRESCA may be regarded as special cases of the general relations assumed in this work.

The idea of integral formulation of the kinematical and statical relations for the continuous body is presented in section 9. Here we introduce a family of space-time regular subregions of the considered region. The tensor fields defined on this family are represented by the corresponding space-time tensor functions.

The construction of the dual spaces determined by the constitutive law is presented in section 10. For the sake of simplicity the derivations are restricted here to the continuous deformations of the body. Analogous construction for the discontinuous deformations is given in section 14.

Finally, using the concepts introduced before, we present an original formulation of the initial-boundary value problem (where only continuous deformations are admitted) in section 11. This formulation leads directly to the corresponding minimum principle, which is established in section 12. The problems of existence and uniqueness are briefly discussed in section 13.

The approach described in the work can easily be generalized to cover a case when the displacement discontinuities are admitted. The modified formulation of the problem, where the internal slip surface constitutes an additional unknown, is presented in section 14. Now the global functions and the bilinear form, used to construct the minimum principle, include additional surface integral over the slip surface.

Finally the solution of the problem can be obtained by minimization of the appropriate function defined on a class of sufficiently regular internal surfaces. Every internal surface from this class determines the set of kinematically admissible displacement functions in which the subsequent minimization is carried out.

2. REGULAR REGIONS

In this section we shall introduce the concept of a regular region in the space R^3. This concept will be used throughout the work to formulate the kinematical and statical relations for the continuous body. The qualification "regular", explained below, is introduced in order to restrict a variety of all possible sets contained in R^3 to the regions V having the boundary B such that the appropriate volume integrals over V and the surface integrals over B can be defined.

A region V will be called regular if:
(i) V is an open bounded set in the space R^3 having two-dimensional boundary B. The volume $|V|$ of the set V, identified with the Lebesgue integral over V, is finite

$$|V| = \int_V dV < \infty \quad . \tag{2.1}$$

(ii) The Lebesgue surface measure is defined on the boundary B of the set V. The surface measure $|A|$ of an open two-dimensional set $A \subset B$ will be referred to as the area of this set. This property enables us to define the surface integral over the boundary B. In particular we have

$$|A| = \int_A dB \quad . \tag{2.2}$$

(iii) The unit vector $\mathbf{n}$, normal to the boundary B and taken as positive outwardly, is defined almost everywhere (with respect to the surface measure) on B. Namely, the normal unit vector $\mathbf{n}$ is represented by the vector function $\boldsymbol{\nu}(A)$

$$\int_A \mathbf{n}\, dB = \boldsymbol{\nu}(A) \tag{2.3}$$

which is determined by the appropriate orthogonal projection of the set A onto the planes $x_1=0$, $x_2=0$ and $x_3=0$.

Here it is necessary to explain more precisely certain notions appearing in the above definition.

Three-dimensional set V of points $\mathbf{x}$ in the space R^3 is called open if and only if for every point $\mathbf{x}_o \in V$ the set V contains a three-dimensional sphere $K_\rho=[\mathbf{x}:|\mathbf{x}-\mathbf{x}_o|<\rho]$ with the centre $\mathbf{x}_o$ and positive radius ρ.

Condition (i) states that the boundary B of the region V is a two-dimensional set in the space R^3. Roughly speaking this set can be obtained by taking pieces of a plane, deforming them continuously and arranging them in such a way that the resulting boundary is a closed surface, which has no self-intersections. According to condition (iii) the surface B is oriented in the space R^3, i.e. the inner and outer faces of this surface are uniquely determined. This property follows from the assumption that B separates the interior of the region V from the exterior.

Two-dimensional set A of the points $\mathbf{x}$ contained in the surface B is called open if and only if for every $\mathbf{x}_o \in A$ the set A contains the intersection $B \cap K_\rho$ of the surface B and the sphere K_ρ (see Fig.1).

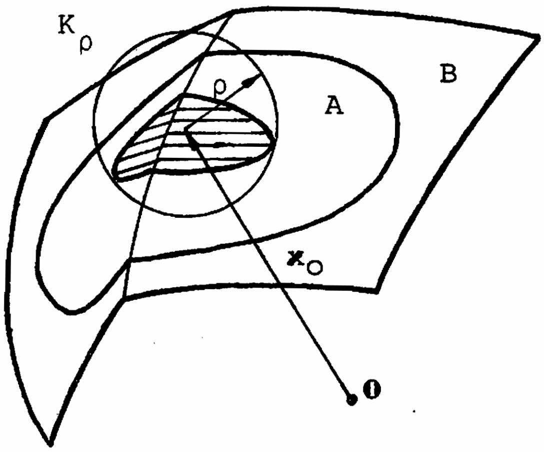

Fig.1. An open set A on two-dimensional surface B in R^3.

The vector function $\mathbf{v}(A)$ defined for every open set A contained in the surface B is introduced here as a primary notion, which will be used to define the surface measure $|A|$ and the normal unit vector $\mathbf{n}$. The construction of this function is presented below.

Let us determine the first component $\nu_1(A)$ of the function $\boldsymbol{\nu}(A)$. We decompose the set A into two disjoint parts: A^+ and A^- as follows: the point $\mathbf{x}=[x_1,x_2,x_3]\in A$ belongs to the set A^+ if there exists a positive real number c_o such that the segment $[x_1-c,x_2,x_3]$, $0<c\leq c_o$ is contained in V (see Fig.2). If this condition is not satisfied then $\mathbf{x}$ belongs to A^-.

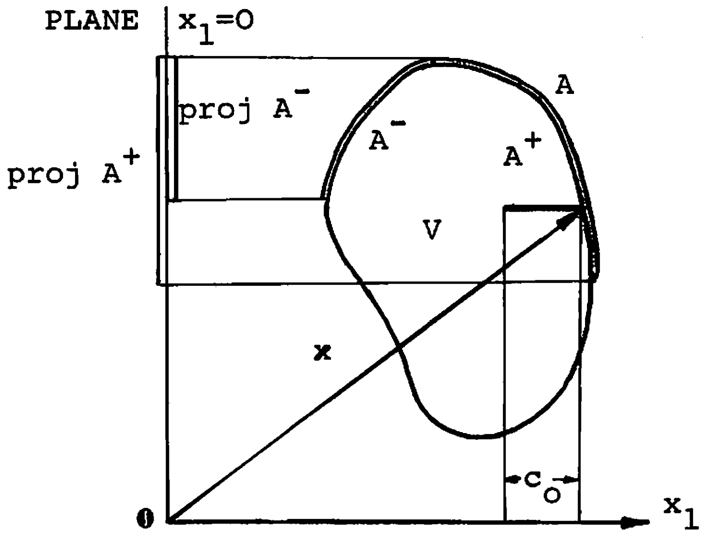

Fig.2. The orthogonal projection of the open set A contained in the boundary B of the region V onto the plane $x_1=0$ in the space R^3.

Assuming that the orthogonal projections proj A^+ and proj A^- onto the plane $x_1=0$ are measurable (in the sense of the Lebesgue measure in the space R^2) we have

$$\nu_1(A) = |\text{proj } A^+|-|\text{proj } A^-| \qquad (2.4)$$

where $|\text{proj } A^+|$ denotes the area of the set proj A^+.

Analogously, repeating the above construction for the axes x_2 and x_3 we obtain the remaining components $\nu_2(A)$ and $\nu_3(A)$ of the vector $\boldsymbol{\nu}(A)$.

To determine the surface measure $|A|$ of the open set A we shall use the family of all sequences of disjoint open sets $A_1,A_2,\ldots$ which are contained in A. The measure $|A|$ is defined as the upper limit of the function $|\boldsymbol{\nu}(A_1)|+|\boldsymbol{\nu}(A_2)|+\ldots$ defined for every sequence from this family

$$|A| = \sup[|\boldsymbol{\nu}(A_1)|+|\boldsymbol{\nu}(A_2)|+\ldots : A_1, A_2, \ldots \subset A, A_i \cap A_j = \emptyset \;\; i \neq j] \qquad (2.5)$$

where $|\boldsymbol{\nu}(A)|$ denotes the length of the vector $\boldsymbol{\nu}(A)$ in R^3.

It should be noted that the positive number $|A|$ can be obtained as the limit of a sequence of the functions corresponding to an ascending sequence of the subdivisions of A into open sets. Such ascending sequence is constructed by consecutive subdivisions of the open sets into smaller parts. The corresponding sequence of functions is non-decreasing, what implies the existence of the limit.

Indeed, taking into account the additivity of the vector function $\boldsymbol{\nu}(A)$,i.e.

$$\boldsymbol{\nu}(A_1)+\boldsymbol{\nu}(A_2) = \boldsymbol{\nu}(A_1 \cup A_2) \qquad (2.6)$$

for arbitrary disjoint open sets $A_1, A_2 \subset B$, we obtain

$$|\boldsymbol{\nu}(A_1)|+|\boldsymbol{\nu}(A_2)| \geq |\boldsymbol{\nu}(A_1 \cup A_2)|. \qquad (2.7)$$

The vector function $\boldsymbol{\nu}(A)$ and the resulting surface measure $|A|$ determine on B the normal unit vector $\mathbf{n}$. Namely, every function $\mathbf{n}(\mathbf{x})$ mapping the points $\mathbf{x}$ from B into unit vectors $\mathbf{n}$, which satisfies the integral relation

$$\int_A \mathbf{n} \, dB = \boldsymbol{\nu}(A) \qquad (2.8)$$

for every open set $A \subset B$ will be called the function of unit vectors normal to the boundary B.

It should be noted that the above definition does not ensure the uniqueness of the function $\mathbf{n}$ on the boundary B. More precisely, there exists a class of functions $\mathbf{n}$ defined on B satisfying the relation (2.8). Two arbitrary functions from this class are equal almost everywhere on B with respect to the surface measure (they may differ in a set of measure zero).

Making use of the above definitions we can prove the following identity

$$\int_B \mathbf{x} \otimes \mathbf{n} \, dB = |V| \, \mathbf{1} \tag{2.9}$$

where the vector $\mathbf{x}$ denotes the coordinates of a point on the boundary B of the regular region V, the symbol $\otimes$ denotes the tensor multiplication of two vectors

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \otimes \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} x_1n_1 & x_1n_2 & x_1n_3 \\ x_2n_1 & x_2n_2 & x_2n_3 \\ x_3n_1 & x_3n_2 & x_3n_3 \end{bmatrix} \tag{2.10}$$

and $\mathbf{1}$ is the unit tensor

$$\mathbf{1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{2.11}$$

Indeed, integrating arbitrary diagonal component of the tensor function $\mathbf{x}\otimes\mathbf{n}$, for example x_1n_1, over the surface B we obtain the volume $|V|$. This result follows from the fact that the integral over B may be presented as the integral over the orthogonal projection $B_{(1)}$ of the region V onto the plane $x_1=0$ (see Fig.3)

$$\int_B x_1n_1 \, dB = \int_{B_{(1)}} \ell(x_2,x_3) \, dB_{(1)} = |V| \tag{2.12}$$

where $\ell(x_2,x_3)$ denotes the total length of the intersection of the line $[c,x_2,x_3], -\infty<c<\infty$ and the region V. The infinitesimal surface $dB_{(1)}$ is here defined as the orthogonal projection of the surface $dB^+\subset B$ (or the corresponding surface $dB^-\subset B$) onto the plane $x_1=0$

$$dB_{(1)} = \nu_1(dB^+) = -\nu_1(dB^-) . \tag{2.13}$$

Integrating arbitrary non-diagonal component of the tensor function $\mathbf{x}\otimes\mathbf{n}$, for example x_2n_1, we obtain

$$\int_B x_2 n_1 \, dB = \int_{B_{(1)}} x_2 \, [\nu_1(dB^+) + \nu_1(dB^-)] = 0. \qquad (2.14)$$

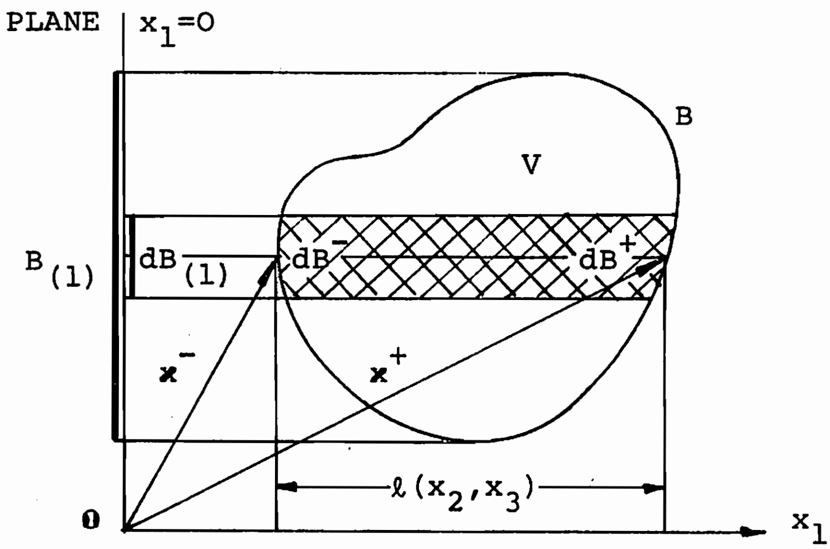

Fig.3. Geometrical interpretation of the identity (2.9).

In the sequel we shall also use the concept of regular region of volume zero in the space R^3. Such region V can be obtained by continuous deformation of arbitrary regular region in such a manner that the outer face of its boundary remains outside, the limit of the volume $|V|$ is equal to zero and the final surface B is measurable and has the normal unit vector (in the sense presented above). It follows from the above definition that the region V of volume zero has the surface measure determined by the measure of the corresponding set A^+ (or A^-) on its boundary B (see Fig.4).

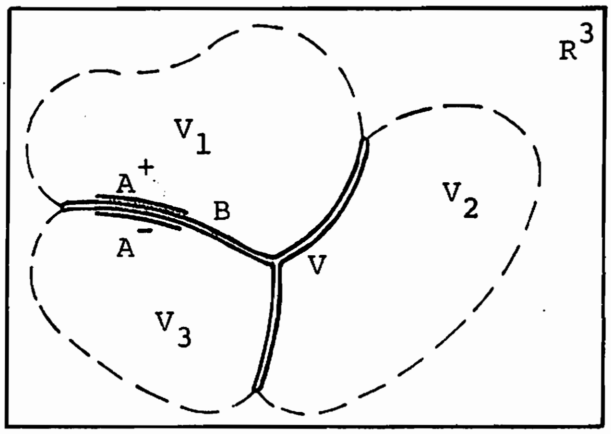

Fig.4. Region V of volume zero in the space R^3. It may be identified with the interface between disjoint regular regions.

3. CONTINUOUS AND DISCONTINUOUS DEFORMATION

Let us consider three-dimensional connected body in the space R^3, which occupies the closure V_o of a regular region. The boundary of the region V_o is denoted by B_o.

The region V_o will be referred to as the initial configuration or the reference configuration of the considered body. The deformation of the body is described in terms of the displacement function $\mathbf{u}$: $V_o \rightarrow R^3$, which maps every point $\mathbf{x} = [x_1, x_2, x_3]$ from V_o into a vector $\mathbf{u} = [u_1, u_2, u_3]$ from the space R^3. The displacement function u determines the transformation of the initial configuration V_o into the actual (or deformed) configuration $\tilde{V}_o$, which is defined as the set of all points $\tilde{\mathbf{x}} = \mathbf{x} + \mathbf{u}(\mathbf{x})$ such that $\mathbf{x} \in V_o$.

We introduce a class of the displacement functions which satisfy the following requirements:

(i) The displacement function $\mathbf{u}$ determines a one-to-one mapping $V_o \Leftrightarrow \tilde{V}_o$ of the region V_o onto the set $\tilde{V}_o \in R^3$.

(ii) The mappings $V_o \rightarrow \tilde{V}_o$ and $\tilde{V}_o \rightarrow V_o$ both map open sets onto open sets, i.e. the image $\tilde{V}$ of arbitrary open set V contained in V_o is an open set contained in $\tilde{V}_o$ and the inverse image V of arbitrary open set $\tilde{V} \in \tilde{V}_o$ is an open set contained in V_o.

(iii) The mappings $V_o \rightarrow \tilde{V}_o$ and $\tilde{V}_o \rightarrow V_o$ both map regular regions onto regular regions (see Fig.5)

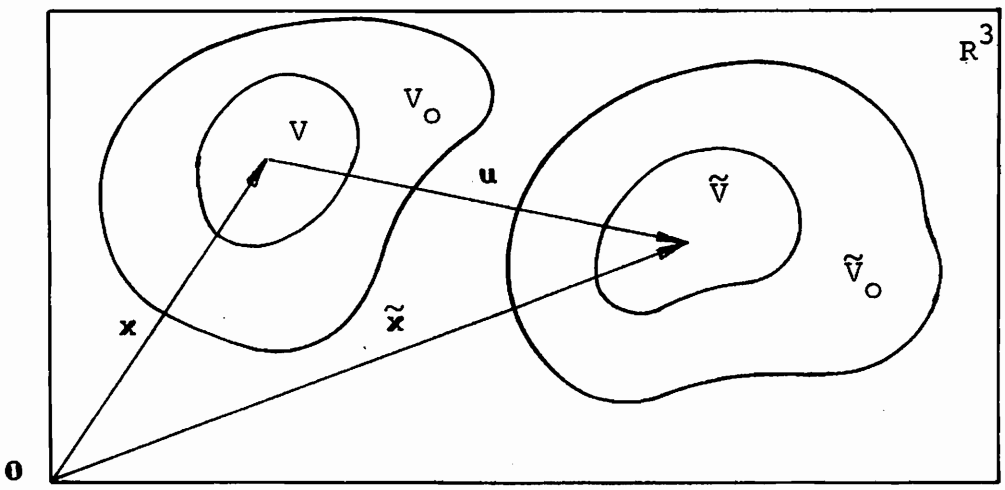

Fig.5.Continuous deformation of the region V_o.

It should be noted that the requirements (i),(ii) imply the continuity of the deformation. This property ensures the compatiblility of the deformed body and excludes a possibility of appearance of the slip surfaces in the considered body.

Due to the property (iii) one can define appropriate surface integrals over the boundary of the regular region both in the reference and the actual configurations of the body. These surface integrals are used throughout the work to formulate the kinematical and statical relations for the considered body.

In order to admit the surfaces of the displacement discontinuity we modify the conditions (i),(ii),(iii) assuming that they concern the points and open sets which do not intersect certain regular region V_ε of zero volume contained in V_o. It is assumed that the region V_ε, called the slip surface or the internal surface of the displacement discontinuity, is transformed during the deformation of the body into a regular region $\tilde{V}_\varepsilon$ of volume zero (see Fig.6).

It should be noted that the modified set of the requirements, which is less restrictive than the original one, also assures the compatibility of the deformed body.

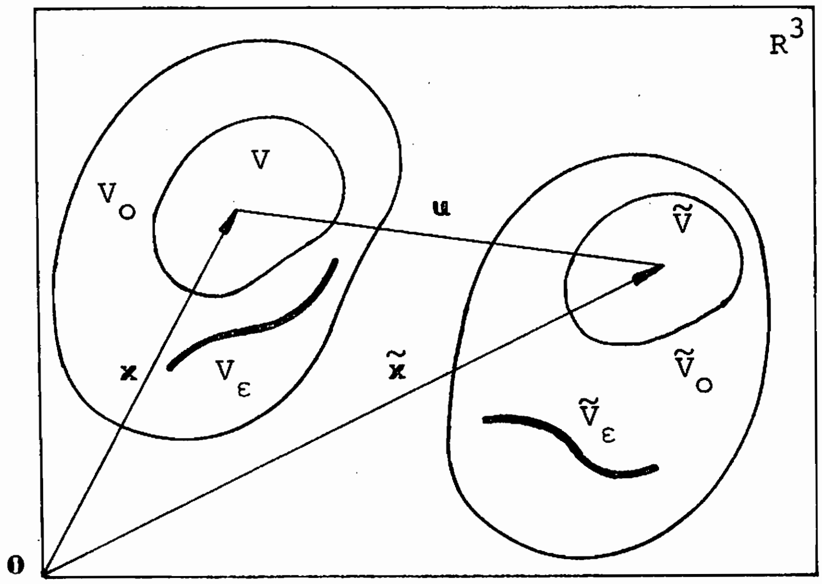

Fig.6. Discontinuous deformation of the region V_o.

4. SECTIONALLY HOMOGENEOUS DEFORMATION

In section 5 we shall introduce the idea of strain and stress based on the assumption of homogeneous deformation of the regular region. The purpose of this section is to show that there exists a wide class of continuous deformations of the body V_o resulting from the homogeneous deformation of its regular subregions. We shall present here one possible method of decomposition of V_o into regular subregions and we shall describe a compatible deformation of the body in terms of the homogeneous deformation of the subregions.

We introduce a regular region U composed of a number of open tetrahedrons $U_i, i=1,2,..,i_o$ in such a way that:
(i) Every tetrahedron U_i has positive volume $|U_i|$.
(ii) Admissible connections of two distinct tetrahedrons are: common side (triangle), common edge (segment of the straight line), common corner (point) or no connection. The set of all points, in which the tetrahedrons have their corners, will be denoted by $\mathbf{x}_{(k)}, k=1,2,..,k_o$ and referred to as the set of nodes.
(iii) The considered region V_o is contained in U in such a manner that every tetrahedron intersects V_o (see Fig.7)

$$V_o \subset U \quad \text{where} \quad U = \bar{U}_1 \cup \bar{U}_2 \cup \ldots \cup \bar{U}_{i_o} \tag{4.1}$$

$$V_o \cap U_i \neq \emptyset \quad \text{for} \quad i=1,2,..,i_o \tag{4.2}$$

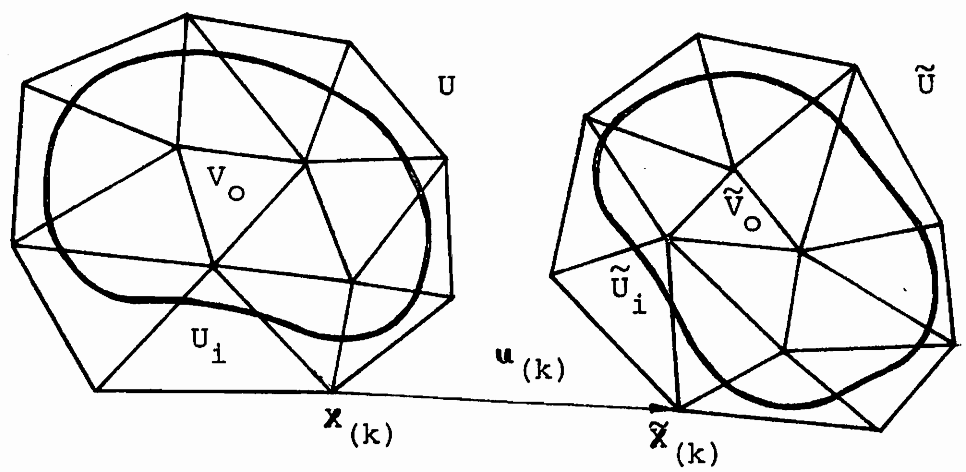

Fig.7. Sectionally homogeneous deformation of body V_o.

The construction of the region U determines a decomposition of the body V_o into the finite number of the regular subregions $V_i, i=1,2,..,i_o$ defined by

$$V_i = U_i \cap V_o \quad \text{for } i=1,2,..,i_o \tag{4.3}$$

Indeed, every region V_i is regular as it is either a tetrahedron (internal subregion) or the intersection of a tetrahedron with the regular region V_o (boundary subregion). It follows from the above construction that these subregions are disjoint and the sum of their closures is equal to V_o.

Let the deformation of the auxiliary region U be determined by the displacements $\mathbf{u}_{(k)}, k=1,2,..,k_o$ of the nodal points $\mathbf{X}_{(k)}, k=1,2,..,k_o$ as follows:
(i) For every tetrahedron U_i the corresponding nodes $\mathbf{X}_{(k)}$, $\mathbf{X}_{(l)}, \mathbf{X}_{(m)}, \mathbf{X}_{(n)}$ do not change their relative orientation in the deformed configuration (see Fig.8)

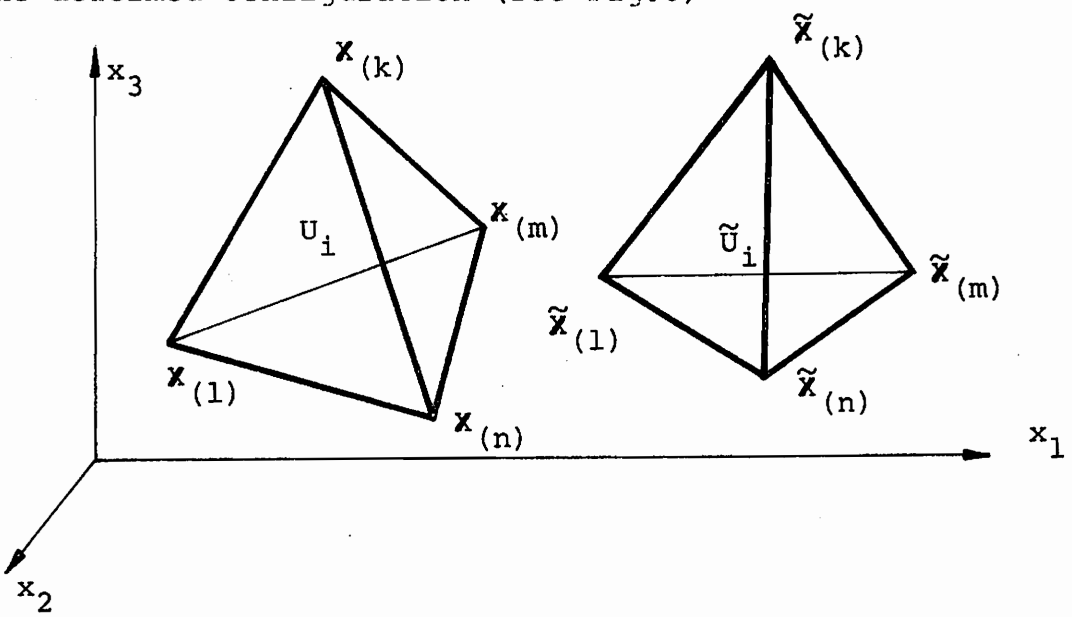

Fig.8. Homogeneous deformation of the tetrahedron U_i.

This requirement can be written in the form of the inequality

$$\det \begin{bmatrix} 1 & \tilde{X}_{1(k)} & \tilde{X}_{2(k)} & \tilde{X}_{3(k)} \\ 1 & \tilde{X}_{1(l)} & \tilde{X}_{2(l)} & \tilde{X}_{3(l)} \\ 1 & \tilde{X}_{1(m)} & \tilde{X}_{2(m)} & \tilde{X}_{3(m)} \\ 1 & \tilde{X}_{1(n)} & \tilde{X}_{2(n)} & \tilde{X}_{3(n)} \end{bmatrix} > 0 \tag{4.4}$$

provided that the initial orientation of the nodes (presented in Fig.8) is such that

$$\det \begin{bmatrix} 1 & X_{1(k)} & X_{2(k)} & X_{3(k)} \\ 1 & X_{1(l)} & X_{2(l)} & X_{3(l)} \\ 1 & X_{1(m)} & X_{2(m)} & X_{3(m)} \\ 1 & X_{1(n)} & X_{2(n)} & X_{3(n)} \end{bmatrix} = \frac{1}{6} |U_i| > 0 \quad . \tag{4.5}$$

(ii) The displacement of a point $\mathbf{x}$ from the tetrahedron U_i is expressed by the linear form

$$\mathbf{u} = \mathbf{u}^o + \mathbf{g}(\mathbf{x}-\mathbf{x}^o) \tag{4.6}$$

where $\mathbf{u}^o$ is the displacement of the centre of volume $\mathbf{x}^o$ of this tetrahedron

$$\mathbf{x}^o = \frac{1}{4} (\mathbf{X}_{(k)}+\mathbf{X}_{(l)}+\mathbf{X}_{(m)}+\mathbf{X}_{(n)}) \tag{4.7}$$

$$\mathbf{u}^o = \frac{1}{4} (\mathbf{u}_{(k)}+\mathbf{u}_{(l)}+\mathbf{u}_{(m)}+\mathbf{u}_{(n)}) \tag{4.8}$$

and the tensor $\mathbf{g}$, called the displacement gradient, can be obtained from the system of linear equations

$$\begin{bmatrix} \Delta X_{1(k)} & \Delta X_{2(k)} & \Delta X_{3(k)} \\ \Delta X_{1(l)} & \Delta X_{2(l)} & \Delta X_{3(l)} \\ \Delta X_{1(m)} & \Delta X_{2(m)} & \Delta X_{3(m)} \end{bmatrix} \begin{bmatrix} g_{11} & g_{21} & g_{31} \\ g_{12} & g_{22} & g_{32} \\ g_{13} & g_{23} & g_{33} \end{bmatrix} = \begin{bmatrix} \Delta u_{1(k)} & \Delta u_{2(k)} & \Delta u_{3(k)} \\ \Delta u_{1(l)} & \Delta u_{2(l)} & \Delta u_{3(l)} \\ \Delta u_{1(m)} & \Delta u_{2(m)} & \Delta u_{3(m)} \end{bmatrix} \tag{4.9}$$

where $\Delta \mathbf{X}_{(k)} = \mathbf{X}_{(k)} - \mathbf{x}^o$ and $\Delta \mathbf{u}_{(k)} = \mathbf{u}_{(k)} - \mathbf{u}^o$.

(iii) The boundary of the deformed region $\tilde{U}$ does not intersects itself.

The deformation described above will be called sectionally homogeneous. From the definition it transforms every tetrahedron U_i into tetrahedron $\tilde{U}_i$. The homogeneous deformation of every tetrahedron is represented by the appropriate displacement gradient $\mathbf{g}$. One can easily show that such deformation satisfies the requirements listed in section 3 for the continuous displacement function. The same requirements are satisfied for the considered region V_o as it is a regular region contained in U.

The concept of sectionally homogeneous deformation of the body constitutes the basis for the description of the material behaviour presented in sections 5,6 and 7. The constitutive law is formulated there for an elementary subregion (in general infinitesimaly small) subjected to the homogeneous deformation.

5. STRAIN AND STRESS TENSORS

Let us consider a regular region V subjected to the homogeneous deformation (4.6) represented by the displacement gradient $\mathbf{g}$. Then arbitrary point $\mathbf{x}$ from V will be transformed into a point $\tilde{\mathbf{x}}$ determined by

$$\tilde{\mathbf{x}}-\tilde{\mathbf{x}}^o = (\mathbf{1}+\mathbf{g})(\mathbf{x}-\mathbf{x}^o) \qquad (5.1)$$

where $\tilde{\mathbf{x}}^o = \mathbf{x}^o+\mathbf{u}^o$ and $\mathbf{u}^o$ is the displacement of the point $\mathbf{x}^o$.

It follows from the orthogonal decomposition theorem that the tensor $\mathbf{1}+\mathbf{g}$ can be uniquely decomposed into a symmetric tensor $\mathbf{1}+\mathbf{h}$ and the orthogonal tensor $\mathbf{r}$

$$\mathbf{1}+\mathbf{g} = (\mathbf{1}+\mathbf{h})\mathbf{r} \quad . \qquad (5.2)$$

Here we recall that the tensor $\mathbf{h}$ is called symmetric if $\mathbf{h}^T=\mathbf{h}$, i.e.

$$\begin{vmatrix} h_{11} & h_{21} & h_{31} \\ h_{12} & h_{22} & h_{32} \\ h_{13} & h_{23} & h_{33} \end{vmatrix} = \begin{vmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{vmatrix} \qquad (5.3)$$

and the tensor $\mathbf{r}$ is orthogonal if $\mathbf{r}^T\mathbf{r} = \mathbf{1}$,i.e.

$$\begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (5.4)$$

The tensor $\mathbf{r}$ may be interpreted as the rigid rotation of the region V with respect to the point $\mathbf{x}^o$. Indeed, the transformation $\mathbf{x}'-\mathbf{x}^o = \mathbf{r}(\mathbf{x}-\mathbf{x}^o)$ does not change the distance between arbitrary points of the region V.

The tensor $\mathbf{1}+\mathbf{h}$ determines the deformation of the tensor V. The transformation $\mathbf{x}''-\mathbf{x}^o = (\mathbf{1}+\mathbf{h})(\mathbf{x}'-\mathbf{x}^o)$ maps the sphere $|\mathbf{x}'-\mathbf{x}^o|=\rho>0$ into an ellipsoid in such a way that the point $\mathbf{x}'$ on the principal axis of the ellipsoid moves along this axis (see Fig.9)

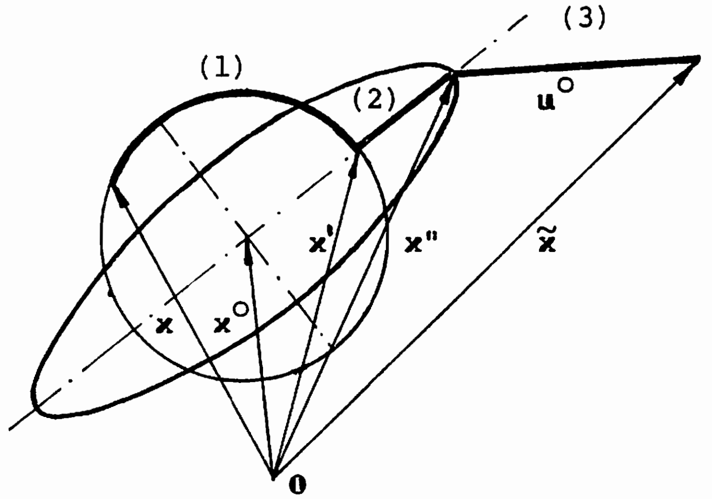

Fig.9. Trajectory of a point $\mathbf{x}$ during the rigid rotation (1), the deformation (2) and the rigid translation (3).

The vector $\mathbf{u}^o$ represents the rigid translation of the body. It should be emphasized that the tensor $\mathbf{h}$ does not depend on the choice of the reference point $\mathbf{x}^o$.

Let us consider a particular homogeneous deformation of the region V described in terms of the scalar parameter λ, which increases monotonously from 0 to 1

$$\tilde{\mathbf{x}}-\tilde{\mathbf{x}}^o = [\mathbf{1}+\mathbf{h}'+\lambda(\mathbf{h}''-\mathbf{h}')]\mathbf{r}(\mathbf{x}-\mathbf{x}^o) \tag{5.5}$$

where $\mathbf{h}'$ and $\mathbf{h}''$ are prescribed symmetric tensors and $\mathbf{r}$ is prescribed orthogonal tensor. For the sake of simplicity we assume that the reference point $\mathbf{x}^o$ is fixed along the considered deformation path, i.e. that $\tilde{\mathbf{x}}^o = \mathbf{x}^o$.

According to the formula (5.5) the considered path starts (λ=0) from the configuration determined by the tensor $(\mathbf{1}+\mathbf{h}')\mathbf{r}$ and finally assumes (λ=1) the configuration determined by $(\mathbf{1}+\mathbf{h}'')\mathbf{r}$. The rigid rotation tensor $\mathbf{r}$ remains constant along this path. The trajectory $\tilde{\mathbf{x}}(\lambda)$ of a point $\mathbf{x}$ during the considered deformation takes the form of the segment of the straight line (see Fig.10).

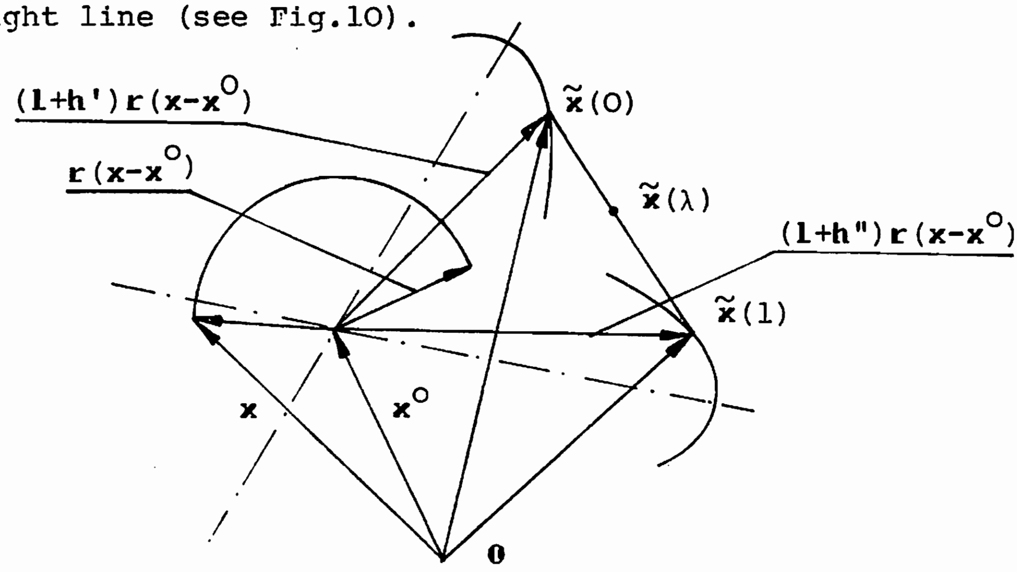

Fig.10. Trajectory $\mathbf{x}(\lambda)$, $0\le\lambda\le 1$.

Let $\tilde{V}(\lambda)$ denote the actual configuration of the region V corresponding to the parameter λ. The increment $d\mathbf{u}$ of the displacement vector $\mathbf{u}$ corresponding to the point $\mathbf{x}$ is expressed by

$$d\mathbf{u} = d\tilde{\mathbf{x}} = d\lambda\ (\mathbf{h}''-\mathbf{h}')\mathbf{r} \quad . \tag{5.6}$$

We introduce the symmetric tensor $\boldsymbol{\varepsilon}$, which satisfies the equation

$$\exp \boldsymbol{\varepsilon} = \mathbf{1}+\mathbf{h}'+\lambda(\mathbf{h}''-\mathbf{h}') \tag{5.7}$$

where the exponential function is defined by

$$\exp \boldsymbol{\varepsilon} = \mathbf{1} + \boldsymbol{\varepsilon} + \frac{1}{2!}\boldsymbol{\varepsilon}\boldsymbol{\varepsilon} + \frac{1}{3!}\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}\boldsymbol{\varepsilon} +\dots \tag{5.8}$$

and the logarithmic function, inverse to the exponent, by

$$\ln(\mathbf{1}+\mathbf{h}) = \mathbf{h} - \frac{1}{2}\mathbf{hh} + \frac{1}{3}\mathbf{hhh} - \ldots \tag{5.9}$$

Making use of the definition (5.7) of the tensor $\boldsymbol{\varepsilon}$ and the following property of the exponential function

$$d(\exp \boldsymbol{\varepsilon}) = d\boldsymbol{\varepsilon} \exp \boldsymbol{\varepsilon} \tag{5.10}$$

we can express the relation (5.6) in the form

$$d\mathbf{u} = d\boldsymbol{\varepsilon} \ (\tilde{\mathbf{x}}(\lambda)-\mathbf{x}^{o}) \quad . \tag{5.11}$$

Indeed, substituting (5.5) into (5.7) we obtain

$$d\mathbf{u} = d\boldsymbol{\varepsilon} \exp \boldsymbol{\varepsilon} \ \mathbf{r}(\mathbf{x}-\mathbf{x}^{o}) = d(\exp \boldsymbol{\varepsilon})\mathbf{r}(\mathbf{x}-\mathbf{x}^{o}) =$$

$$= d\lambda \ (\mathbf{h}''-\mathbf{h}')\mathbf{r}(\mathbf{x}-\mathbf{x}^{o}) \quad . \tag{5.12}$$

Let us denote by $\tilde{\mathbf{f}}$ the surface force applied to the boundary $\tilde{B}(\lambda)$ of the regular region $\tilde{V}(\lambda)$ and by $\tilde{\mathbf{b}}$ the body force defined in $\tilde{V}(\lambda)$. Assuming that $\tilde{\mathbf{f}}$ and $\tilde{\mathbf{b}}$ are integrable we can express the increment dw of the external work supplied to the considered region in the form

$$dw = \int_{\tilde{B}(\lambda)} \tilde{\mathbf{f}}\cdot d\mathbf{u} \ d\tilde{B} + \int_{\tilde{V}(\lambda)} \tilde{\mathbf{b}}\cdot d\mathbf{u} \ d\tilde{V} \quad . \tag{5.13}$$

Substituting the relation (5.11) into (5.13) we obtain

$$dw = d\boldsymbol{\varepsilon}\cdot\left(\int_{\tilde{B}(\lambda)} \tilde{\mathbf{f}} \otimes (\tilde{\mathbf{x}}-\mathbf{x}^{o})d\tilde{B} + \int_{\tilde{V}(\lambda)} \tilde{\mathbf{b}} \otimes (\tilde{\mathbf{x}}-\mathbf{x}^{o})d\tilde{V}\right) \quad . \tag{5.14}$$

Assuming that the tensor $\boldsymbol{\sigma}$ defined as

$$\boldsymbol{\sigma} = \frac{1}{|V|}\left(\int_{\tilde{B}(\lambda)} \tilde{\mathbf{f}} \otimes (\tilde{\mathbf{x}}-\mathbf{x}^{o})d\tilde{B} + \int_{\tilde{V}(\lambda)} \tilde{\mathbf{b}} \otimes (\tilde{\mathbf{x}}-\mathbf{x}^{o})d\tilde{V}\right) \tag{5.15}$$

remains constant along the deformation path we obtain the external work w supplied to the considered region expressed in terms of $\boldsymbol{\varepsilon}' = \ln(\mathbf{1}+\mathbf{h}')$ and $\boldsymbol{\varepsilon}'' = \ln(\mathbf{1}+\mathbf{h}'')$.

$$w = \int_{\lambda=0}^{1} dw = |V| \ \boldsymbol{\sigma} \cdot (\boldsymbol{\varepsilon}'' - \boldsymbol{\varepsilon}') \quad . \tag{5.16}$$

Here it is convenient to discuss the properties of the tensor $\boldsymbol{\sigma}$ introduced above. Let us observe that if the forces applied to the region $\tilde{V}(\lambda)$ are in equilibrium

$$\int_{\tilde{B}(\lambda)} \tilde{\mathbf{f}} \, d\tilde{B} + \int_{\tilde{V}(\lambda)} \tilde{\mathbf{b}} \, d\tilde{V} = \mathbf{0} \tag{5.17}$$

then the tensor $\boldsymbol{\sigma}$ does not depend on the choice of the reference point $\mathbf{x}^o$. If the moments (with respect to $\mathbf{x}^o$) applied to the region $\tilde{V}(\lambda)$ are in equilibrium

$$\mathbf{m} = \int_{\tilde{B}(\lambda)} \tilde{\mathbf{f}} \times (\tilde{\mathbf{x}} - \mathbf{x}^o) \, dB + \int_{\tilde{V}(\lambda)} \tilde{\mathbf{b}} \times (\tilde{\mathbf{x}} - \mathbf{x}^o) \, d\tilde{V} = \mathbf{0} \tag{5.18}$$

where symbol $\times$ denotes the vector product, then the tensor $\boldsymbol{\sigma}$ is symmetric. Indeed, the skew-symmetric part of $\boldsymbol{\sigma}$ can be expressed in the form

$$\frac{1}{2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}^T) = \frac{1}{2} \frac{1}{|V|} \begin{bmatrix} 0 & m_3 & -m_2 \\ -m_3 & 0 & m_1 \\ m_2 & -m_1 & 0 \end{bmatrix} \tag{5.19}$$

where m_1, m_2, m_3 are the components of the moment of force $\mathbf{m}$.

Now let us consider the increment dw of the external work supplied to the considered region during the rigid rotation with respect to the reference point $\mathbf{x}^o$. Now the increment $d\mathbf{u}$ can be expressed by

$$d\mathbf{u} = d\mathbf{r} \ (\tilde{\mathbf{x}} - \mathbf{x}^o) \tag{5.20}$$

where skew-symmetric tensor $d\mathbf{r}$ is determined by the rotation vector $d\boldsymbol{\omega}$

$$d\mathbf{r} = \begin{bmatrix} 0 & d\omega_3 & -d\omega_2 \\ -d\omega_3 & 0 & d\omega_1 \\ d\omega_2 & -d\omega_1 & 0 \end{bmatrix} \tag{5.21}$$

Substituting (5.20) into (5.13) and making use of the definition (5.15) we obtain

$$dw = |V| \, \boldsymbol{\sigma} \cdot d\mathbf{r} = 0 \quad . \tag{5.22}$$

Indeed, the scalar product of the symmetric tensor $\boldsymbol{\sigma}$ and the skew-symmetric tensor $d\mathbf{r}$ always vanishes.

Similarly for the rigid translation of the considered region, assuming the equilibrium of forces (5.17), we obtain directly from (5.13) that the external work supplied to the region is equal to zero.

Finally we conclude that the external work w supplied to the material contained in a regular region V subjected to the homogeneous deformation composed of finite or infinite number of pure deformations (5.5), rigid rotations and rigid translations depends only on the initial and the final states of deformation

$$w = |V| \, \boldsymbol{\sigma} \cdot (\boldsymbol{\varepsilon}'' - \boldsymbol{\varepsilon}') \tag{5.23}$$

provided that the region V is in equilibrium and that the tensor $\boldsymbol{\sigma}$ remains constant along the deformation path. The symmetric tensors $\boldsymbol{\varepsilon}'$ and $\boldsymbol{\varepsilon}''$ are uniquely determined from the orthogonal decompositions: $\exp \boldsymbol{\varepsilon}'\mathbf{r}' = \mathbf{1} + \mathbf{g}'$ and $\exp \boldsymbol{\varepsilon}''\mathbf{r}'' = \mathbf{1} + \mathbf{g}''$ of the initial and final displacement gradients $\mathbf{g}'$ and $\mathbf{g}''$.

The tensor $\boldsymbol{\varepsilon}$, appearing in the above derivations, will be called the <u>strain tensor</u> representing the deformation of the regular region V. Taking into account the linear form (4.6) of the displacement function $\mathbf{u}$ and the identity (2.9) we can express the relation between the deformation and the displacement in the integral form

$$|V| \, (\exp \boldsymbol{\varepsilon} \, \mathbf{r} - \mathbf{1}) = \int_B \mathbf{u} \otimes \mathbf{n} \, dB \quad . \tag{5.24}$$

The tensor σ, defined by the relation (5.15), will be called the stress tensor corresponding to the region V. It is convenient to express the stress tensor by appropriate integrals in the reference configuration. Namely, we introduce the reference boundary forces **f** defined on B and the reference body forces **b** defined in V, which satisfy the relations

$$\int_{A} \mathbf{f}\, dB = \int_{\tilde{A}} \tilde{\mathbf{f}}\, d\tilde{B} \tag{5.25}$$

for every corresponding open sets $A \subset B$ and $\tilde{A} \subset \tilde{B}$ (Fig.11) and

$$\int_{U} \mathbf{b}\, dV = \int_{\tilde{U}} \tilde{\mathbf{b}}\, d\tilde{V} \tag{5.26}$$

for every corresponding regular regions $U \subset V$ and $\tilde{U} \subset \tilde{V}$.

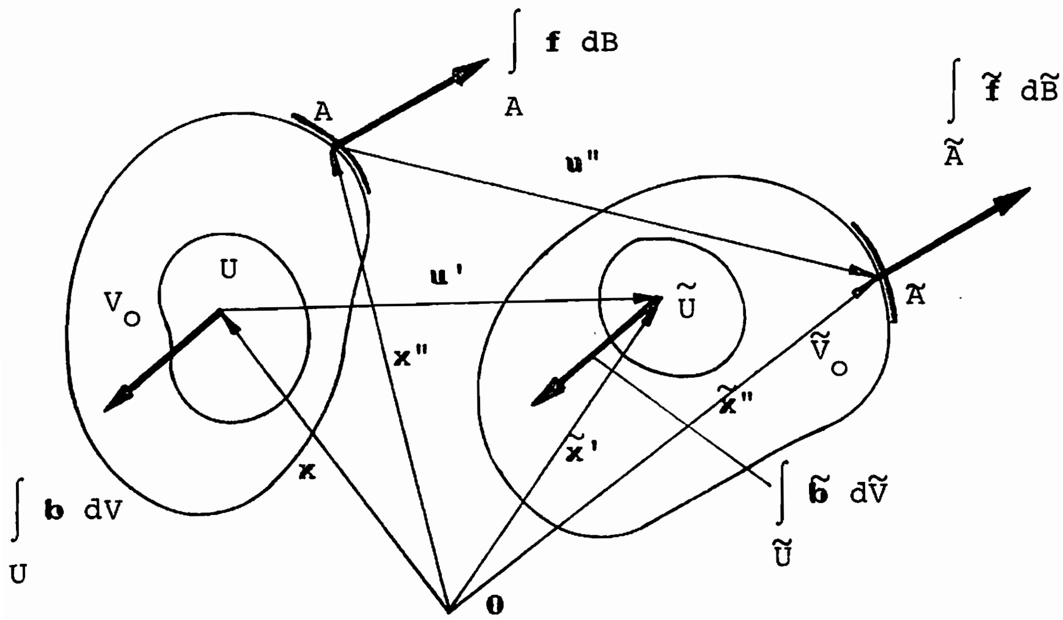

Fig.11. Concept of the reference forces.

Now we can express the stress tensor by

$$\sigma = \frac{1}{|V|}\left(\int_{B} \mathbf{f} \otimes (\mathbf{x}+\mathbf{u})\, dB + \int_{V} \mathbf{b} \times (\mathbf{x}+\mathbf{u})\, dV \right) \tag{5.27}$$

6. INTERNAL PARAMETERS

We shall consider the homogeneous deformation of the regular region V. We assume that the material contained in V is a homogeneous structure. For simplicity the initial configuration of the body V is assumed to coincide with the natural state of the material. Consequently we shall refer the deformation of the body to the initial configuration.

We shall describe the loading and the deformation of the body V in the system of coordinates $\tilde{x}'_1, \tilde{x}'_2, \tilde{x}'_3$, which is obtained by the rotation $\mathbf{r}$ of the system x_1, x_2, x_3 about the origin. The rotation tensor $\mathbf{r}$ results (5.2) from the actual displacement gradient $\mathbf{g}$.

It follows from the above construction that in the transformed coordinate system the body V is subjected only to pure deformation and uniform translation. The strain and stress tensors $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$, defined in the system x_1, x_2, x_3 , are expressed in the form $\mathbf{r}^T\boldsymbol{\varepsilon}\mathbf{r}$ and $\mathbf{r}^T\boldsymbol{\sigma}\mathbf{r}$ in the rotating system $\tilde{x}'_1, \tilde{x}'_2, \tilde{x}'_3$ (see Fig.12)

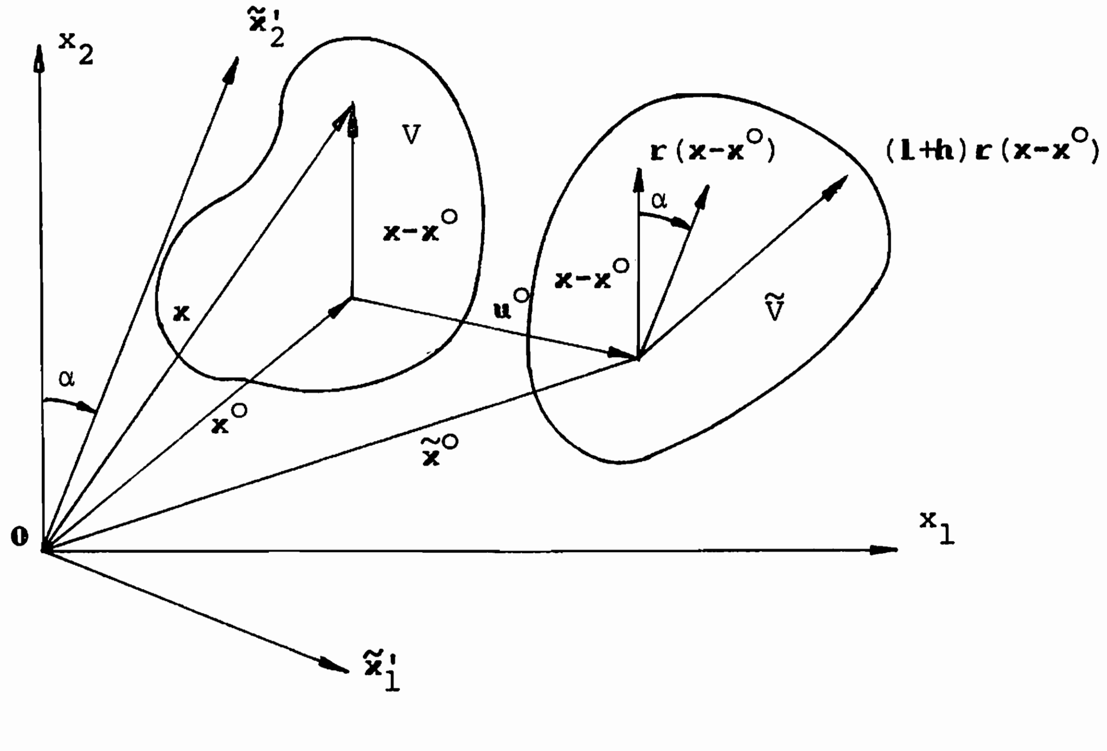

Fig.12. The rotating system of coordinates determined by the deformation of the body V.

To describe the material behaviour we shall use the concept of the internal parameters. Namely, we introduce the notions of the <u>internal stress</u> $\mathbf{s} = [\mathbf{s}_1,\mathbf{s}_2,..,\mathbf{s}_n]$ and the <u>internal strain</u> $\mathbf{e} = [\mathbf{e}_1,\mathbf{e}_2,..,\mathbf{e}_n]$, where the components $\mathbf{s}_i,\mathbf{e}_i, i=1,2,..,n$ are symmetric tensors defined in the coordinate system $\tilde{x}_1',\tilde{x}_2',\tilde{x}_3'$ rotating with the body V. Here we assume that these tensors describe the forces in the material structure and the internal deformations of this structure, respectively.

Here it is convenient to introduce the notation of the tensor spaces which are used throughout the work. The space of all symmetric tensors will be denoted by T_S and the space of all orthogonal tensors by T_R. We shall denote by T and T^* the dual spaces of all internal stress tensors $\mathbf{s}$ and all internal strain tensors $\mathbf{e}$, respectively. These spaces are provided with the scalar product defined by

$$\mathbf{s}\cdot\mathbf{e} = \mathbf{s}_1\cdot\mathbf{e}_1 + \mathbf{s}_2\cdot\mathbf{e}_2 +\ldots+ \mathbf{s}_n\cdot\mathbf{e}_n \tag{6.1}$$

which represents the work of the forces $\mathbf{s}$ on the displacement $\mathbf{e}$. The above definition implies that, from the mathematical point of view, the space T may be identified with T^* and they are the Hilbert spaces. Formally we can present these spaces as the cartesian product of n spaces T_S, i.e. $T = T^* = T_S\times T_S\times\ldots\times T_S$.

We assume that the relation between the internal stress $\mathbf{s}$, representing the forces in the material, and the stress $\boldsymbol{\sigma}$, representing the external loading of the material, is established by the following <u>equilibrium condition</u>

$$\mathbf{s}_1 = \mathbf{r}^T\boldsymbol{\sigma}\mathbf{r} \quad . \tag{6.2}$$

It is also assumed here that the external loading is uniformly distributed in the region V, i.e. that the tensor $\boldsymbol{\sigma}$ corresponds not only to the region V but also for every regular subregion V' contained in V. In general such condition is supposed to be satisfied for an infinitesimal regular region V.

The relation between the internal strain $\mathbf{e}$ and the strain $\boldsymbol{\varepsilon}$ follows from the <u>balance of mechanical energy</u>

$$\mathbf{s}\cdot d\mathbf{e} = \boldsymbol{\sigma}\cdot d\boldsymbol{\varepsilon} \qquad \text{for every } \mathbf{s}\in T \tag{6.3}$$

where $\boldsymbol{\sigma}\cdot d\boldsymbol{\varepsilon}$ is the energy supplied to the material (see section 5) and $\mathbf{s}\cdot d\mathbf{e}$ is the energy absorbed by the material structure. The balance of energy (6.3) in conjunction with the equlibrium condition (6.2) implies that

$$d\mathbf{e}_1 = \mathbf{r}^T d\boldsymbol{\varepsilon}\, \mathbf{r} \quad , \; d\mathbf{e}_2 = \mathbf{0} \;, \ldots , \; d\mathbf{e}_n = \mathbf{0} \quad . \tag{6.4}$$

Now we shall assume that the energy $\mathbf{s}\cdot d\mathbf{e}$ absorbed in the material can be decomposed into the energy $\mathbf{s}\cdot d\mathbf{e}^e$ stored in the material and the energy $\mathbf{s}\cdot d\mathbf{e}^p$ dissipated in the material

$$\mathbf{s}\cdot d\mathbf{e} = \mathbf{s}\cdot d\mathbf{e}^e + \mathbf{s}\cdot d\mathbf{e}^p \qquad \text{for every } \mathbf{s}\in T \quad . \tag{6.5}$$

The tensors of <u>elastic strain</u> $\mathbf{e}^e\in T^*$ and <u>plastic strain</u> $\mathbf{e}^p\in T^*$ introduced here characterize the internal state of the displacements in the material structure.

It should be noted that the elastic and plastic strains represent, by the definition, the fractions of the mechanical energy absorbed in the material and they have no direct geometrical meaning on the macroscopic level.

Assuming that in the initial configuration the elastic and plastic strains vanish and taking into account (6.4) we obtain

$$\mathbf{e}_1^e + \mathbf{e}_1^p = \mathbf{r}^T\boldsymbol{\varepsilon}\mathbf{r} \;, \; \mathbf{e}_2^e + \mathbf{e}_2^p = \mathbf{0} \;, \; \ldots \;, \; \mathbf{e}_n^e + \mathbf{e}_n^p = \mathbf{0} \quad . \tag{6.6}$$

Finally the material behaviour will be described in terms of the internal stress tensor in the form

$$\mathbf{s} = [\mathbf{r}^T\boldsymbol{\sigma}\mathbf{r}, \mathbf{s}_2, \ldots , \mathbf{s}_n] \in T \tag{6.7}$$

and the following strain tensors

$$\mathbf{e} = [\mathbf{r}^T \boldsymbol{\varepsilon} \mathbf{r}, \mathbf{0}, \ldots, \mathbf{0}] \in T^*$$

$$\mathbf{e}^e = [\mathbf{e}_1^e, \mathbf{e}_2^e, \ldots, \mathbf{e}_n^e] \in T^* \qquad (6.8)$$

$$\mathbf{e}^p = [\mathbf{e}_1^p, \mathbf{e}_2^p, \ldots, \mathbf{e}_n^p] \in T^*$$

where

$$\mathbf{e} = \mathbf{e}^e + \mathbf{e}^p . \qquad (6.9)$$

One can try to give a physical interpretation of the internal stress introduced in this section. Namely, we shall suppose that the considered material structure consists of a number of elementary components. The elementary component is assumed to be either purely elastic or purely viscoplastic, i.e. the entire mechanical energy absorbed by the component is either stored or dissipated. All components are assumed to be uniformly distributed in the considered regular region V, i.e. they interpenetrate each other.

Let the hypothetical material structure consist of m distinct elementary components and let the loading of an individual component of the structure be represented by the symmetric tensor $\boldsymbol{\tau}_i$, where $1 \le i \le m$. In other words the real surface force $\mathbf{f}_i$ applied to the i-th component of the structure on the plane cross section, determined by the normal unit vector $\mathbf{n}$, is expressed by

$$\mathbf{f}_i = \boldsymbol{\tau}_i \mathbf{n} . \qquad (6.10)$$

To describe the material structure we shall postulate the particular interactions between the components. For example one can connect a part of the components in series

$$\boldsymbol{\tau}_1 = \boldsymbol{\tau}_2 = \ldots = \boldsymbol{\tau}_k = \mathbf{s}_1 \quad \text{where} \quad k = m-n > 0 \qquad (6.11)$$

and the remaining parts of the components in parallel

$$\tau_{k+1} = s_1 - (s_2 + s_3 + .. + s_{m-k})$$

$$\tau_{k+2} = s_2 \;, \; \tau_{k+3} = s_3 \;, \ldots, \; \tau_m = s_{m-k} \;. \qquad (6.12)$$

The equations (6.11),(6.12) may be interpreted as the equilibrium equations for the elements of the structure, by analogy to the one-dimensional system presented in Fig.13, where τ_i,i=1,2,..,m denote the axial forces in the elements. The set $s_1, s_2, .., s_n$ of auxiliary forces, independent of each other, uniquely determines the state of the forces in the system.

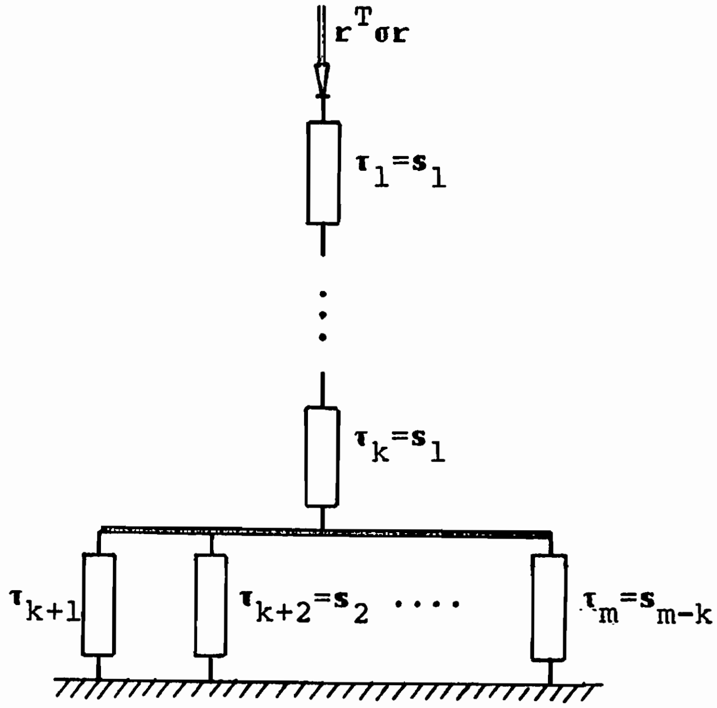

Fig.13. One-dimensional analogy of the material structure.

Using the above analogy we may interprete the internal stress tensor $s = [s_1, s_2, .., s_n]$ as a system of auxiliary stress tensors, independent of each other, which in conjunction with postulated internal connections (6.11),(6.12) between the components of the material structure uniquely determines the loading τ_i,i=1,2,..,m of these components.

Similarly, using the one-dimensional analogy, one can interprete the equilibrium of the internal and external forces $s_1 = r^T \sigma r$. It should be noted that the internal forces can not be measured on the macroscopic level and the equations (6.11),(6.12) are not sufficient to determine these forces unless the structure is statically determined.

Following the idea presented above one can construct a variety of material structures postulating distinct systems of elementary components connected in parallel and in series. Some typical material structures, used in the literature, are presented (with the formalism proposed in this section) in section 8.

7. CONSTITUTIVE RELATIONS

The elastic behaviour of the material structure is described by the relation between the internal stress tensor s and the elastic strain tensor $\mathbf{e}^e$

$$\mathbf{s} = A_e \, \mathbf{e}^e \quad . \tag{7.1}$$

Taking into account the assumption that the tensor $\mathbf{e}^e$ represents the deformation of the elastic components of the material structure (see section 6) we formulate the basic postulate of the elastic behaviour as follows:
The mapping $A_e : T^* \to T$ (not necessarily one-to-one) has a potential $\psi^* : T^* \to R^1$, i.e. there exists the scalar function $\psi^*(\mathbf{e}^e)$ such that the integral over a path C, connecting the points $\mathbf{e}'$ and $\mathbf{e}''$ in the space T^*, is expressed by

$$\int_C d\mathbf{e}^e \cdot A_e \mathbf{e}^e = \psi^*(\mathbf{e}'') - \psi^*(\mathbf{e}') \quad . \tag{7.2}$$

In other words the mechanical energy stored in the material structure is uniquely determined by the actual strain tensor $\mathbf{e}^e$.

The potential ψ^* introduced above is called the free energy function. We shall assume that the function ψ^* is lower-semicontinuous, convex and it attains its minimum equal to zero at the origin of the space T^*.

Here it is convenient to recall the mathematical concepts used in this section. The function ψ^*, defined in T^*, is called lower-semicontinuous if for every real number c the set of all tensors $\mathbf{e}^e$ from T^* satisfying $\psi^*(\mathbf{e}^e) \leq c$ is closed. The function ψ^* is called convex if

$$c_1 \ \psi^*(\mathbf{e}') + c_2 \ \psi^*(\mathbf{e}'') \geq \psi^*(c_1\mathbf{e}' + c_2\mathbf{e}'') \qquad (7.3)$$

for arbitrary $\mathbf{e}', \mathbf{e}'' \in T^*$ and $0 \leq c_1 \leq 1$, where $c_2 = 1 - c_1$. The tensor $\mathbf{s}$ is called the subgradient of the function ψ^* at the point $\mathbf{e}^e$ if the inequality

$$(\mathbf{e} - \mathbf{e}^e) \cdot \mathbf{s} \leq \psi^*(\mathbf{e}) - \psi^*(\mathbf{e}^e) \qquad (7.4)$$

holds true for every tensor $\mathbf{e}$ from the space T^*. The set of all subgradients at the point $\mathbf{e}^e$ is called the subdifferential of ψ^* at $\mathbf{e}^e$ and is denoted by $\partial\psi^*(\mathbf{e}^e)$.

It has been shown in [15] that if the function ψ^* is lower-semicontinuous and convex then the mapping A_e is the subgradient of ψ^* in the space T^*

$$\mathbf{s} \in \partial\psi^*(\mathbf{e}^e) \quad . \qquad (7.5)$$

In the particular case the free energy function is determined by a set $\mathbf{L}_1, \mathbf{L}_2, .., \mathbf{L}_n$ of positive definite tensors of fourth rank

$$\psi^*(\mathbf{e}^e) = \frac{1}{2} \ (\mathbf{e}_1^e \cdot \mathbf{L}_1 \mathbf{e}_1^e + \mathbf{e}_2^e \cdot \mathbf{L}_2 \mathbf{e}_2^e + \ldots + \mathbf{e}_n^e \cdot \mathbf{L}_n \mathbf{e}_n^e) \quad . \qquad (7.6)$$

The quadratic form (7.6) can be also denoted by

$$\psi^*(\mathbf{e}^e) = \frac{1}{2} \ \mathbf{e}^e \cdot \mathbf{L} \mathbf{e}^e \qquad (7.7)$$

where $\mathbf{L}$ denotes the diagonal tensor composed of the set of elastic moduli $\mathbf{L}_1, \mathbf{L}_2, .., \mathbf{L}_n$. The constitutive relation (7.5) takes now the linear form

$$\mathbf{s} = \mathbf{L} \ \mathbf{e}^e \qquad (7.8)$$

equivalent, by the definition, to the set of relations

$$\mathbf{s}_1 = \mathbf{L}_1\mathbf{e}_1^e \ , \ \mathbf{s}_2 = \mathbf{L}_2\mathbf{e}_2^e \ , \ \dots \ , \ \mathbf{s}_n = \mathbf{L}_n\mathbf{e}_n^e \qquad (7.9)$$

which may be referred to as the generalized Hooke's law.

The viscoelastic behaviour of the material structure will be described in this work by the relation between the rate of the plastic strain tensor $\mathbf{e}^p$ and the internal stress tensor $\mathbf{s}$

$$\dot{\mathbf{e}}^p = A_p \ \mathbf{s} \quad . \qquad (7.10)$$

Following the idea presented in [3],[4],[5],[6] we shall assume that the function A_p is the subgradient of the dissipation potential φ prescribed in the space T of all internal stress tensors $\mathbf{s}$

$$\dot{\mathbf{e}}^p \in \partial\varphi(\mathbf{s}) \ . \qquad (7.11)$$

It is assumed that the function φ is lower-semicontinuous, convex and it attains its minimum equal to zero at the origin of the space T.

The constitutive relation (7.11) postulated above can describe both viscous and plastic behaviour of the material. Some particular cases of such visco-plastic behaviour is presented in section 8.

It should be noted that the postulated constitutive relation ensures that the dissipation rate $\mathbf{s}\cdot\dot{\mathbf{e}}^p$ is always non-negative. Indeed, taking into account the definition (7.4) of the subgradient and the assumption $\varphi(\mathbf{s})\geq\varphi(\mathbf{0})$ we obtain

$$\mathbf{s}\cdot\dot{\mathbf{e}}^p \geq \varphi(\mathbf{s})-\varphi(\mathbf{0}) \geq 0 \ . \qquad (7.12)$$

The constitutive relations (7.5) and (7.11) can be also presented in the form

$$\mathbf{e}^e \in \partial\psi(\mathbf{s}) \qquad (7.13)$$

$$\mathbf{s} \in \partial\varphi^*(\dot{\mathbf{e}}^p) \tag{7.14}$$

where the polar function ψ of the free energy function is defined in the space T by

$$\psi(\mathbf{s}) = \sup_{\mathbf{e}\in T^*} [\mathbf{s}\cdot\mathbf{e}-\psi^*(\mathbf{e})] \tag{7.15}$$

and the polar function φ^* of the dissipation potential is defined in the space T^* by

$$\varphi^*(\dot{\mathbf{e}}^p) = \sup_{\mathbf{s}\in T} [\mathbf{s}\cdot\dot{\mathbf{e}}^p-\varphi(\mathbf{s})] \quad . \tag{7.16}$$

8. EXAMPLES OF ELASTIC-VISCOPLASTIC MATERIALS

We shall discuss two particular material structures represented by one-dimensional models given in Fig.14. The springs in the picture represent the elastic components of the material structure characterized by Young moduli $\mathbf{L}_1$ and $\mathbf{L}_2$, the slide represents the perfectly plastic component characterized by the yield stress θ and the dashpot represents the viscous component characterized by the positive constant μ.

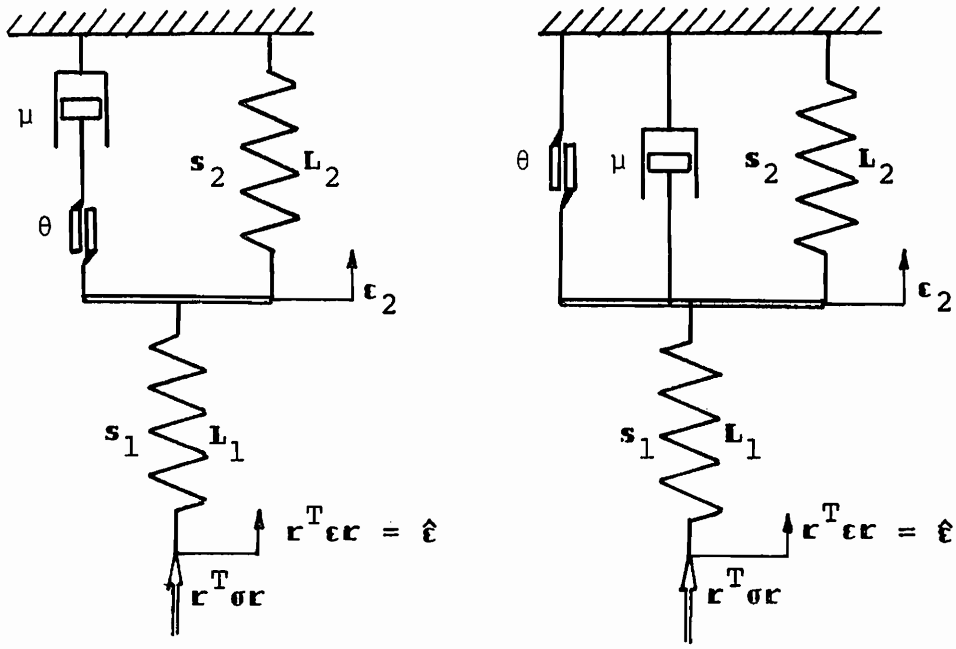

Fig.14. One-dimensional models of elastic-viscoplastic materials.

The external loading of the system is represented by the tensor $\mathbf{r}^T\boldsymbol{\sigma}\mathbf{r}$ and the deformation of the system is represented by the tensor $\mathbf{r}^T\boldsymbol{\varepsilon}\mathbf{r} = \hat{\boldsymbol{\varepsilon}}$.

Let us consider the first system presented in Fig.14. Let the internal stresses $\mathbf{s}_1$ and $\mathbf{s}_2$ be interpreted as the forces in the springs $\mathbf{L}_1$ and $\mathbf{L}_2$, respectively, and let $\boldsymbol{\varepsilon}_2$ represent the deformation of the spring $\mathbf{L}_2$. Making use of the results presented in section 6 we can express the internal stress and strains in terms of the tensors $\mathbf{s}_1, \mathbf{s}_2$, $\hat{\boldsymbol{\varepsilon}}, \boldsymbol{\varepsilon}_2$ as follows

$$\mathbf{s} = [\mathbf{s}_1, \mathbf{s}_2] \qquad \mathbf{e}^e = [\hat{\boldsymbol{\varepsilon}} - \boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_2]$$
$$\mathbf{e} = [\hat{\boldsymbol{\varepsilon}}, \mathbf{0}] \qquad \mathbf{e}^p = [\boldsymbol{\varepsilon}_2, -\boldsymbol{\varepsilon}_2] \, . \tag{8.1}$$

The free energy function is defined as the quadratic form

$$\psi^*(\mathbf{e}^e) = \frac{1}{2}(\hat{\boldsymbol{\varepsilon}} - \boldsymbol{\varepsilon}_2) \cdot \mathbf{L}_1 (\hat{\boldsymbol{\varepsilon}} - \boldsymbol{\varepsilon}_2) + \frac{1}{2}\boldsymbol{\varepsilon}_2 \cdot \mathbf{L}_2 \boldsymbol{\varepsilon}_2 \, . \tag{8.2}$$

Because the plastic component and the viscous component are connected in series the dissipation potential is expressed as the sum of the plastic and viscous part

$$\varphi(\mathbf{s}) = \varphi_p(\mathbf{s}) + \varphi_v(\mathbf{s}) \, . \tag{8.3}$$

The viscous potential is defined as the quadratic form

$$\varphi_v(\mathbf{s}) = \frac{1}{2}(\mathbf{s}_1 - \mathbf{s}_2) \cdot \mu (\mathbf{s}_1 - \mathbf{s}_2) \tag{8.4}$$

and the plastic potential is defined as the <u>indicator function</u> determined by the convex region E contained in the space T of all **internal** stress tensors

$$\varphi_p(\mathbf{s}) = \begin{cases} 0 & \text{if } \mathbf{s} \in E \\ +\infty & \text{if } \mathbf{s} \notin E \, . \end{cases} \tag{8.5}$$

The set E, referred to as the plastically admissible region, is defined either with Huber-Von Mises criterion

$$E = [\mathbf{s}:\ \mathbf{s}\in T\ ,\ |\mathbf{s}_1'-\mathbf{s}_2'|\leq\theta] \tag{8.6}$$

where $\mathbf{s}_1'$ and $\mathbf{s}_2'$ denote the deviators of the tensors $\mathbf{s}_1$ and $\mathbf{s}_2$, or with Tresca criterion

$$E = [\mathbf{s}:\ \mathbf{s}\in T\ ,\ |(\mathbf{s}_1'-\mathbf{s}_2')_{(m)}-(\mathbf{s}_1'-\mathbf{s}_2')_{(n)}|\leq\theta \text{ for } m,n=1,2,3] \tag{8.7}$$

where $(\mathbf{s}_1'-\mathbf{s}_2')_{(m)}$,m=1,2,3 denotes the principal components of the tensor $\mathbf{s}_1'-\mathbf{s}_2'$.

The second system presented in Fig.14 is characterized by parallel connection of the viscous and plastic components. In this case the dissipation potential is determined by its polar function expressed in the form

$$\varphi^*(\dot{\mathbf{e}}^p) = \varphi_p^*(\dot{\mathbf{e}}^p) + \varphi_v^*(\dot{\mathbf{e}}^p)\ . \tag{8.8}$$

The plastic behaviour of the material structure determined by the region E in the form (8.6) or (8.7) may be identified with the kinematic work hardening model commonly used in applied plasticity, where the tensor $\mathbf{s}_2$ is interpreted as the centre of the <u>yield surface</u> in the space of all tensors $\mathbf{s}_1$. According to such interpretation the yield surface is subjected to rigid translation during the plastic process.

More general model, frequently used in plasticity, admits the isotropic expansion of the yield surface in the space of all tensors $\mathbf{s}_1$. One can easily extend the description given in this section to include this general model of work hardening materials. Namely, introducing additional component of the internal stress, which represents an isotropic expansion of the yield surface

$$\mathbf{s}_3 = \pi\ \mathbf{1} \tag{8.9}$$

one can express the material behaviour in terms of

$$\mathbf{s} = [\mathbf{s}_1,\mathbf{s}_2,\mathbf{s}_3]\ ,\quad \mathbf{e}^e = [\hat{\boldsymbol{\varepsilon}}-\boldsymbol{\varepsilon}_2,\boldsymbol{\varepsilon}_2,\boldsymbol{\varepsilon}_3]$$

$$\mathbf{e} = [\hat{\boldsymbol{\varepsilon}},\mathbf{0},\mathbf{0}] \quad , \quad \mathbf{e}^p = [\boldsymbol{\varepsilon}_2,-\boldsymbol{\varepsilon}_2,-\boldsymbol{\varepsilon}_3] \tag{8.10}$$

where the strain tensor $\boldsymbol{\varepsilon}_3$, corresponding to $\mathbf{s}_3$, takes the form

$$\boldsymbol{\varepsilon}_3 = \frac{1}{3}\,\omega\,\mathbf{1} \quad . \tag{8.11}$$

Now the free energy function is defined by

$$\psi^*(\mathbf{e}^e) = \frac{1}{2}(\hat{\boldsymbol{\varepsilon}}-\boldsymbol{\varepsilon}_2)\cdot\mathbf{L}_1(\hat{\boldsymbol{\varepsilon}}-\boldsymbol{\varepsilon}_2)+\frac{1}{2}\boldsymbol{\varepsilon}_2\cdot\mathbf{L}_2\boldsymbol{\varepsilon}_2+\frac{1}{2}\boldsymbol{\varepsilon}_3\cdot L_3\boldsymbol{\varepsilon}_3 \tag{8.12}$$

where L_3 is a positive constant, and the definitions (8.6) and (8.7) assume the form

$$E = [\mathbf{s}\colon \mathbf{s}\in T \ , \ |\mathbf{s}_1'-\mathbf{s}_2'|\le\theta+\pi] \tag{8.13}$$

$$E = [\mathbf{s}\colon \mathbf{s}\in T \ , \ |(\mathbf{s}_1'-\mathbf{s}_2')_{(m)}-(\mathbf{s}_1'-\mathbf{s}_2')_{(n)}|\le\theta+\pi \text{ for } m,n=1,2,3] \ . \tag{8.14}$$

It follows from the constitutive relations (7.5),(7.11) that for such material model we have

$$\pi = \frac{1}{3}\,L_3\,\omega \tag{8.15}$$

$$\omega = \int_0^t |\dot{\boldsymbol{\varepsilon}}_2|\,dt \quad . \tag{8.16}$$

9. STRAIN AND STRESS FIELDS

In order to formulate the initial-boundary value problem for the elastic-viscoplastic body we shall use the concept of the strain and stress fields defined in four-dimensional space-time region. Namely, we shall assume that the state of deformation of the considered body is determined by a tensor function $\mathbf{e}(\mathbf{x},t)\colon \mathbf{V}_o\to T^*$, which maps every point $\mathbf{x},t$ from the space-time region $\mathbf{V}_o = V_o\times[0,t_o]$ into a strain tensor $\mathbf{e}$ from the space T^*.

The relation between the strain function $\mathbf{e}(\mathbf{x},t)$ and the strain tensor $\mathbf{e}$ (which was introduced in section 5 as a primary concept) corresponding to the space-time region $\mathbf{V} = V \times [t_1,t_2]$ is here established by

$$\mathbf{e}(\mathbf{V}) = \frac{1}{m(\mathbf{V})} \int_V \int_{t_1}^{t_2} \mathbf{e}(x,t)\ h(t)\ dV\ dt \tag{9.1}$$

where V is a regular region contained in V_o, $0 \le t_1 < t_2 \le t_o$, h(t) is a non-negative decreasing function prescribed in the considered time interval $[0,t_o]$ and the volume measure of the region $\mathbf{V}$ is defined by

$$m(\mathbf{V}) = |V| \int_{t_1}^{t_2} h(t)\ dt\ . \tag{9.2}$$

The family of all space-time regions $\mathbf{V}$ of positive measure $m(\mathbf{V})$ contained in $\mathbf{V}_o$ will be denoted by M. In section 14 we shall also use the family M^o of all space-time regions $\mathbf{V}$ of volume measure zero, i.e. the regions $\mathbf{V} = V \times [t_1,t_2]$ where V is a regular region of volume zero contained in V_o.

Now we can introduce the concept of the strain field $\mathbf{e}(\mathbf{V})$: $M \to T^*$. This field is defined as the mapping, which establishes the strain tensor $\mathbf{e}$ from T^* for every space-time region $\mathbf{V}$ from the family M. It is assumed here that there exists an integrable function $\mathbf{e}(\mathbf{x},t)$ which determines the strain field $\mathbf{e}$ with the relation (9.1).

Let us denote by C the set of all continuous (see section 3) displacement functions $\mathbf{u}(\mathbf{x},t)$: $\mathbf{V}_o \to R^3$ defined in the considered space-time region.

We shall express the relation between the displacement function and the internal strain function in the integral form. Namely, the strain function

$$\mathbf{e}(\mathbf{x},t) = [\mathbf{r}^T(\mathbf{x},t)\boldsymbol{\varepsilon}(\mathbf{x},t)\mathbf{r}(\mathbf{x},t),\mathbf{0},..,\mathbf{0}] \tag{9.3}$$

determined by the functions $\boldsymbol{\varepsilon}(\mathbf{x},t)$: $\mathbf{V}_o \to T_S$ and $\mathbf{r}(x,t)$: $\mathbf{V}_o \to T_R$

will be called continuous and compatible if there exists the displacement function $\mathbf{u}(\mathbf{x},t) \in C$ such that the equation

$$\int_B \int_{t_1}^{t_2} \mathbf{u}(\mathbf{x},t) \times \mathbf{n}(\mathbf{x}) \; h(t)\,dB\;dt = \int_V \int_{t_1}^{t_2} [\exp\, \boldsymbol{\varepsilon}(\mathbf{x},t)\mathbf{r}(\mathbf{x},t) - \mathbf{1}]h(t)\,dV\;dt \tag{9.4}$$

holds true for every region V from the family M. Here B denotes the boundary of the regular region V and $\mathbf{n}(\mathbf{x})$ is the unit vector normal to B.

In the sequel we shall use the simplified notation of the surface and volume integrals in the space-time. According to this notation the equation (9.4) takes the form

$$\int_{\mathbf{B}} \mathbf{u} \otimes \mathbf{n} \; dm_B = \int_{\mathbf{V}} (\exp\, \boldsymbol{\varepsilon}\; \mathbf{r} - \mathbf{1})\; dm \tag{9.5}$$

where $\mathbf{B} = B \times [t_1, t_2]$, m_B is the surface measure defined by

$$m_B(\mathbf{B}) = |B| \int_{t_1}^{t_2} h(t)\; dt \tag{9.6}$$

and m is the volume measure defined by (9.2).

It is assumed in the work that the surface $\mathbf{B}_o = B_o \times [0, t_o]$ of the region $\mathbf{V}_o$ is composed of the measurable part $\mathbf{B}_o^u$, where we prescribe the boundary displacement $\mathbf{u}_o(\mathbf{x},t)$ and the remaining part $\mathbf{B}_o^f$ where we prescribe the boundary force $\mathbf{f}_o(\mathbf{x},t)$

$$\mathbf{B}_o = \mathbf{B}_o^u \cup \mathbf{B}_o^f \; . \tag{9.7}$$

For every fixed continuous compatible strain function $\mathbf{e}(\mathbf{x},t)$ we shall define the family of all statically admissible internal stress functions $\mathbf{s}(\mathbf{x},t)$: $\mathbf{V}_o \to \mathbf{T}$. Let N denote the unit sphere $|\mathbf{x}|=1$ in the space R^3. The stress function

$$\mathbf{s}(\mathbf{x},t) = [\mathbf{r}^T(\mathbf{x},t)\boldsymbol{\sigma}(\mathbf{x},t)\mathbf{r}(\mathbf{x},t), \mathbf{s}_2, .., \mathbf{s}_n] \tag{9.8}$$

will be called statically admissible if there exists the surface force function $\mathbf{f}(\mathbf{n},\mathbf{x},t): N \times V_o \to R^3$ such that the equations

$$\int_B \mathbf{f} \otimes (\mathbf{x}+\mathbf{u}) \; dm_B + \int_V \mathbf{b} \otimes (\mathbf{x}+\mathbf{u}) \; dm = \int_V \boldsymbol{\sigma} \; dm \qquad (9.9)$$

$$\int_B \mathbf{f} \; dm_B + \int_V \mathbf{b} \; dm = \mathbf{0} \qquad (9.10)$$

hold true for every $\mathbf{V}$ from the family M. Here $\boldsymbol{\sigma}(\mathbf{x},t): \mathbf{V}_o \to T_S$ denotes the stress function and $\mathbf{b}(\mathbf{x},t): \mathbf{V}_o \to R^3$ denotes the prescribed body force function in the reference configuration.

The equation (9.4) expresses the compatibility condition for the functions $\boldsymbol{\varepsilon}(\mathbf{x},t)$ and $\mathbf{r}(\mathbf{x},t)$, which represent the deformation of the body. The equation (9.10) expresses the equilibrium of forces for the deformed body. The equilibrium of moments for the deformed body

$$\int_B \mathbf{f} \times (\mathbf{x}+\mathbf{u}) \; dm_B + \int_V \mathbf{b} \times (\mathbf{x}+\mathbf{u}) \; dm = \mathbf{0} \qquad (9.11)$$

is implied by the assumption that the function $\boldsymbol{\sigma}(\mathbf{x},t)$ maps the region $\mathbf{V}_o$ into the space of all symmetric tensors T_S.

The function of the plastic strain rate $\dot{\mathbf{e}}^p(\mathbf{x},t): \mathbf{V}_o \to T^*$ is here introduced as a primary concept. We assume that the plastic strain function is determined by its rate with the relation

$$\mathbf{e}^p(\mathbf{x},t) = \int_0^t \dot{\mathbf{e}}^p(\mathbf{x},t') \; dt' \qquad (9.12)$$

which implies that the initial value of the plastic strain function is assumed to vanish

$$\mathbf{e}^p(\mathbf{x},0) = \mathbf{0} \quad . \qquad (9.13)$$

Now we shall prove that if the prescribed function $h(t)$ has the following property: $h(t_o)=0$ and there exists $c>0$ such that

$$h(t_1)-h(t_2) \geq c \; h(t_1) \; (t_2-t_1) \qquad (9.14)$$

holds true for every $0 \le t_1 < t_2 \le t_o$, then the existence of the integral

$$\int_{\mathbf{V}_o} \dot{\mathbf{e}}^p(\mathbf{x},t)\ dm \qquad (9.15)$$

implies the existence of the integral

$$\int_{\mathbf{V}_o} \mathbf{e}^p(\mathbf{x},t)\ dm \ . \qquad (9.16)$$

Indeed, taking into account the definition of the measure $m(V)$ and integrating (9.15) by parts we obtain

$$\int_{V_o} \int_0^{t_o} \dot{\mathbf{e}}^p(\mathbf{x},t)\ h(t) dV\ dt = \int_{V_o} \int_0^{t_o} \mathbf{e}^p(\mathbf{x},t)\ dh(t)\ dV \ . \qquad (9.17)$$

Using the inequality (9.14) we conclude that the measure dh dV bounds from above the measure dm. Hence the integral (9.16) exists.

It should be noted that the function $h(t) = \exp(-t) - \exp(-t_o)$ satisfies all requirements postulated above.

It is assumed in the sequel that all functions $\mathbf{e}(\mathbf{x},t)$, $\dot{\mathbf{e}}^p(\mathbf{x},t)$, $\mathbf{e}^e(\mathbf{x},t)$, $\mathbf{s}(\mathbf{x},t)$ are integrable in the sense of the integral (9.15). The integrability of the function $\mathbf{e}^p(\mathbf{x},t)$, defined by (9.12), follows from the particular properties of the prescribed function $h(t)$.

10. DUAL SPACES

Let $L^1(\mathbf{V}_o)$ denote the space of all stress tensor functions $\mathbf{s}(\mathbf{x},t)$: $V_o \to T$ which are integrable in the space-time region $\mathbf{V}_o$, i.e.

$$\left| \int_{\mathbf{V}_o} \mathbf{s}\ dm \right| < +\infty \ . \qquad (10.1)$$

We shall use the same notation $L^1(\mathbf{V}_o)$ for the space of all integrable strain tensor functions $\mathbf{e}(\mathbf{x},t)$: $\mathbf{V}_o \to T^*$.

We introduce the notion of the <u>global functions</u> defined in the space $L^1(\mathbf{V}_o)$ by the free energy function $\psi^*(\mathbf{e}^e)$ and the dissipation potential $\varphi(\mathbf{s})$

$$\Psi(\mathbf{s}) = \int_{\mathbf{V}_o} \psi(\mathbf{s}(\mathbf{x},t))\ dm \qquad (10.2)$$

$$\Psi^*(\mathbf{e}^e) = \int_{\mathbf{V}_o} \psi^*(\mathbf{e}^e(\mathbf{x},t))\ dm \qquad (10.3)$$

$$\Phi(\mathbf{s}) = \int_{\mathbf{V}_o} \varphi(\mathbf{s}(\mathbf{x},t))\ dm \qquad (10.4)$$

$$\Phi^*(\dot{\mathbf{e}}^p) = \int_{\mathbf{V}_o} \varphi^*(\dot{\mathbf{e}}^p(\mathbf{x},t))\ dm \ . \qquad (10.5)$$

It follows from the convexity of ψ^* and φ that the global functions defined above exist. Indeed, every such space-time integral can be obtained as the limit of the sequence of approximations corresponding to an ascending sequence of subdivisions of the region $\mathbf{V}_o$ into disjoint subregions. Such sequence is always non-decreasing as

$$\psi\left(\frac{1}{m(\mathbf{V}_1)}\int_{\mathbf{V}_1}\mathbf{s}\ dm\right)m(\mathbf{V}_1)+\psi\left(\frac{1}{m(\mathbf{V}_2)}\int_{\mathbf{V}_2}\mathbf{s}\ dm\right)m(\mathbf{V}_2)\geq\psi\left(\frac{1}{m(\mathbf{V})}\int_{\mathbf{V}}\mathbf{s}\ dm\right)m(\mathbf{V}) \qquad (10.6)$$

for every disjoint $\mathbf{V}_1$ and $\mathbf{V}_2$ contained in $\mathbf{V}_o$, where $\mathbf{V} = \mathbf{V}_1 \cup \mathbf{V}_2$. Hence the limit (finite or infinite) always exists and may be expressed by

$$\Psi(\mathbf{s}) = \sup\left[\sum_{i=1}^{\infty}\psi\left(\frac{1}{m(\mathbf{V}_i)}\int_{V_i}\mathbf{s}\ dm\right)m(\mathbf{V}_i):\ \mathbf{V}_i\subset\mathbf{V}_o, i=1,2,\ldots,\ \mathbf{V}_i\cap\mathbf{V}_j=\emptyset \text{ for } i\neq j\right]. \qquad (10.7)$$

The set of all functions s from $L^1(\mathbf{V}_o)$ for which the global function $\Psi(\mathbf{s})$ is finite is called the <u>effective domain</u> of Ψ and denoted by dom Ψ

$$\text{dom}\ \Psi(\mathbf{s}) = [\mathbf{s}:\ \mathbf{s}\in L^1(\mathbf{V}_o)\ ,\ \Psi(\mathbf{s})<+\infty]\ . \qquad (10.8)$$

The effective domain of the global function Ψ determines the subspace P_e in the space $L^1(\mathbf{V}_o)$. Namely, the stress function $\mathbf{s}$ from $L^1(\mathbf{V}_o)$ belongs to the set P_e if there exists a positive number c such that c $\mathbf{s}$ is contained in dom Ψ

$$P_e = [\mathbf{s}:\ \mathbf{s}\in L^1(\mathbf{V}_o)\ ,\ \text{there exists } c>0 \text{ such that } \Psi(c\mathbf{s})<+\infty]\ . \tag{10.9}$$

Since the effective domain is convex and its interior contains the origin of $L^1(\mathbf{V}_o)$ the set P_e has all properties of the functional subspace.

Similarly the effective domain of the global function Ψ^* determines the subspace P_e^* in the space $L^1(\mathbf{V}_o)$

$$P_e^* = [\mathbf{e}:\ \mathbf{e}\in L^1(\mathbf{V}_o)\ ,\ \text{there exists } c>0 \text{ such that } \Psi^*(c\mathbf{e})<+\infty]\ . \tag{10.10}$$

Now we shall prove the following theorem:
<u>For every stress function</u> $\mathbf{s}$ <u>from the space</u> P_e <u>and every strain function</u> $\mathbf{e}$ <u>from the space</u> P_e^* <u>the bilinear form</u>

$$(\mathbf{s},\mathbf{e}) = \int_{\mathbf{V}_o} \mathbf{s}(\mathbf{x},t)\cdot\mathbf{e}(\mathbf{x},t)\ dm \tag{10.11}$$

<u>exists and is finite</u>.

Proof: For every stress tensor $\mathbf{s}\in T$ and every strain tensor $\mathbf{e}\in T^*$ we have the inequalities

$$\mathbf{s}\cdot\mathbf{e} \leq \psi(\mathbf{s})+\psi^*(\mathbf{e}) \tag{10.12}$$

$$-\mathbf{s}\cdot\mathbf{e} \leq \psi(-\mathbf{s})+\psi^*(\mathbf{e}) \tag{10.13}$$

which follow directly from the definition of the polar function (7.15). Integrating the functions $\mathbf{s}(\mathbf{x},t)$ and $\mathbf{e}(\mathbf{x},t)$ from $L^1(\mathbf{V}_o)$ over the region $\mathbf{V}_o$ and taking into account (10.12),(10.13) we obtain

$$\int_{\mathbf{V}_o} |\mathbf{s}\cdot\mathbf{e}|dm \leq \max[\Psi(\mathbf{s})+\Psi^*(\mathbf{e}),\Psi(-\mathbf{s})+\Psi^*(\mathbf{e})]\ . \tag{10.14}$$

Hence the bilinear form $(\mathbf{s},\mathbf{e})$ exists provided that $\mathbf{e} \in \mathrm{dom}\,\Psi$ and $\mathbf{s} \in Q$, where the set Q is defined by

$$Q = [\mathbf{s}:\ \mathbf{s} \in \mathrm{dom}\,\Psi\ ,\ -\mathbf{s} \in \mathrm{dom}\,\Psi]\ . \qquad (10.15)$$

It follows from the above definition that the set Q absorbs the space P_e, i.e. for every $\mathbf{s} \in P_e$ there exists $c_1>0$ such that $c_1\mathbf{s} \in Q$. Similarly it follows directly from the definition of the space P_e^* that for every $\mathbf{e} \in P_e^*$ there exists $c_2>0$ such that $c_2\mathbf{e} \in \mathrm{dom}\,\Psi^*$. Hence for arbitrary $\mathbf{s} \in P_e$ and $\mathbf{e} \in P_e^*$ we can calculate the bilinear form as

$$(\mathbf{s},\mathbf{e}) = \frac{1}{c_1 c_2}(c_1\mathbf{s},c_2\mathbf{e}) \quad \square \qquad (10.16)$$

The bilinear form $(\mathbf{s},\mathbf{e})$ places the spaces P_e and P_e^* in duality. This duality is supposed to be separating, i.e. two functions $\mathbf{s}'$ and $\mathbf{s}''$ from P_e are identical if and only if $(\mathbf{s}'-\mathbf{s}'',\mathbf{e})=0$ for every $\mathbf{e}$ from P_e^*. Similarly two functions $\mathbf{e}'$ and $\mathbf{e}''$ from P_e^* are identical if and only if $(\mathbf{s},\mathbf{e}'-\mathbf{e}'')=0$ for every $\mathbf{s} \in P_e$.

Following the idea of the free energy function ψ^* and the relation (7.5) we postulate the elastic deformation law

$$\mathbf{s} \in \partial\Psi^*(\mathbf{e}^e) \qquad (10.17)$$

i.e. the internal stress function $\mathbf{s}(\mathbf{x},t) \in P_e$ and the elastic strain function $\mathbf{e}^e(\mathbf{x},t) \in P_e^*$ satisfy the inequality

$$(\mathbf{e}-\mathbf{e}^e,\mathbf{s}) \le \Psi^*(\mathbf{e})-\Psi^*(\mathbf{e}^e) \qquad (10.18)$$

for every $\mathbf{e}(\mathbf{x},t) \in P_e^*$.

Similarly we construct in the space $L^1(\mathbf{V}_o)$ second pair of the dual spaces P_p and P_p^*, determined by the global dissipation potential $\Phi(\mathbf{s})$

$$P_p = [\mathbf{s}:\ \mathbf{s} \in L^1(\mathbf{V}_o)\ ,\ \text{there exists } c>0 \text{ such that } \Phi(c\mathbf{s})<+\infty] \qquad (10.19)$$

$$P_p^* = [\mathbf{e}:\ \mathbf{e}\in L^1(\mathbf{V}_o)\ ,\ \text{there exists } c>0 \text{ such that } \Phi^*(c\mathbf{e})<+\infty] \tag{10.20}$$

In an analogous way, following the relation (7.11) we postulate the plastic flow law in the global form

$$\dot{\mathbf{e}}^p \in \partial\Phi(\mathbf{s}) \tag{10.21}$$

i.e. the internal stress function $\mathbf{s}(x,t)\in P_p$ and the plastic strain rate function $\dot{\mathbf{e}}^p(x,t)\in P_p^*$ satisfy the inequality

$$(\mathbf{s}'-\mathbf{s},\dot{\mathbf{e}}^p) \leq \Phi(\mathbf{s}')-\Phi(\mathbf{s}) \tag{10.22}$$

for every $\mathbf{s}'(\mathbf{x},t)\in P_p$.

The polar functions satisfy the following relations

$$\Psi(\mathbf{s}) = \sup_{\mathbf{e}\in P_e^*} [(\mathbf{s},\mathbf{e})-\Psi^*(\mathbf{e})] \tag{10.23}$$

$$\Phi^*(\dot{\mathbf{e}}^p) = \sup_{\mathbf{s}\in P_p} [(\mathbf{s},\dot{\mathbf{e}}^p)-\Phi(\mathbf{s})] \quad . \tag{10.24}$$

11. INITIAL-BOUNDARY VALUE PROBLEM

Making use of the concepts introduced in previous sections we can formulate the initial-boundary value problem for the elastic-viscoplastic body as follows:
Find the displacement function $\mathbf{u}(\mathbf{x},t):\ \mathbf{V}_o\to R^3$ from the set C and the surface force function $\mathbf{f}(\mathbf{n},\mathbf{x},t):\ N\times\mathbf{V}_o\to R^3$, vanishing at the initial moment

$$\mathbf{u}(\mathbf{x},0) = \mathbf{0} \tag{11.1}$$

$$\mathbf{f}(\mathbf{n},\mathbf{x},0) = \mathbf{0} \tag{11.2}$$

and satisfying the boundary conditions

$$\mathbf{u}(\mathbf{x},t) = \mathbf{u}_o(\mathbf{x},t) \qquad \text{on the boundary } \mathbf{B}_o^u \tag{11.3}$$

$$\mathbf{f}(\mathbf{n},\mathbf{x},t) = \mathbf{f}_o(\mathbf{x},t) \qquad \text{on the boundary } \mathbf{B}_o^f \tag{11.4}$$

such that the following requirements are satisfied:

1. There exist the strain function $\boldsymbol{\varepsilon}(\mathbf{x},t): \mathbf{V}_o \to T_S$ and the rotation function $\mathbf{r}(\mathbf{x},t): \mathbf{V}_o \to T_R$ which satisfy the relation

$$\int_{\mathbf{B}} \mathbf{u} \otimes \mathbf{n} \; dm_B = \int_{\mathbf{V}} (\exp \boldsymbol{\varepsilon} \; \mathbf{r} - \mathbf{1}) \; dm \tag{11.5}$$

for every subregion $\mathbf{V}$ from the family M.

2. There exists the stress function $\boldsymbol{\sigma}(\mathbf{x},t): \mathbf{V}_o \to T_S$ which satisfies the stress-force relation

$$\int_{\mathbf{B}} \mathbf{f} \otimes (\mathbf{x}+\mathbf{u}) \; dm_B + \int_{\mathbf{V}} \mathbf{b} \otimes (\mathbf{x}+\mathbf{u}) \; dm = \int_{\mathbf{V}} \boldsymbol{\sigma} \; dm \tag{11.6}$$

for every subregion $\mathbf{V}$ from the family M.

3. The surface force function $\mathbf{f}(\mathbf{n},\mathbf{x},t)$ and the prescribed body force function $\mathbf{b}(\mathbf{x},t): \mathbf{V}_o \to R^3$ satisfy the equilibrium of forces

$$\int_{\mathbf{B}} \mathbf{f} \; dm_B + \int_{\mathbf{V}} \mathbf{b} \; dm = \mathbf{0} \tag{11.7}$$

and the equilibrium of moments

$$\int_{\mathbf{B}} \mathbf{f} \times (\mathbf{x}+\mathbf{u}) \; dm_B + \int_{\mathbf{V}} \mathbf{b} \times (\mathbf{x}+\mathbf{u}) \; dm = \mathbf{0} \tag{11.8}$$

for every subregion $\mathbf{V}$ from the family M.

4. There exists the plastic strain rate function $\dot{\mathbf{e}}^p(\mathbf{x},t): \mathbf{V}_o \to T^*$ which is plastically admissible

$$\Phi^*(\dot{\mathbf{e}}^p) = \int_{\mathbf{V}_o} \varphi^*(\dot{\mathbf{e}}^p(\mathbf{x},t)) \; dm < +\infty \quad . \tag{11.9}$$

The plastic strain function $\mathbf{e}^p(\mathbf{x},t): \mathbf{V}_o \to T^*$ is determined by the plastic strain rate function

$$\mathbf{e}^p(\mathbf{x},t) = \int_0^t \dot{\mathbf{e}}^p(\mathbf{x},t') \; dt' \tag{11.10}$$

and, according to this definition, is assumed to vanish at the initial moment.

5. There exists the elastic strain function $\mathbf{e}^e(\mathbf{x},t)\colon \mathbf{V}_o \to T^*$ which is elastically admissible

$$\Psi^*(\mathbf{e}^e) = \int_{\mathbf{V}_o} \psi^*(\mathbf{e}^e(\mathbf{x},t))\ dm < +\infty \quad . \qquad (11.11)$$

6. There exists the internal stress function $\mathbf{s}(\mathbf{x},t)\colon \mathbf{V}_o \to T$ which is both elastically and plastically admissible

$$\Phi(\mathbf{s}) = \int_{\mathbf{V}_o} \varphi(\mathbf{s}(\mathbf{x},t))\ dm < +\infty \qquad (11.12)$$

$$\Psi(\mathbf{s}) = \int_{\mathbf{V}_o} \psi(\mathbf{s}(\mathbf{x},t))\ dm < +\infty \quad . \qquad (11.13)$$

7. The internal stress function $\mathbf{s} = [\mathbf{s}_1,\mathbf{s}_2,..,\mathbf{s}_n]$ is in equilibrium with the stress function $\boldsymbol{\sigma}$

$$\mathbf{s}_1(\mathbf{x},t) = \mathbf{r}^T(\mathbf{x},t)\boldsymbol{\sigma}(\mathbf{x},t)\mathbf{r}(\mathbf{x},t) \quad . \qquad (11.14)$$

8. The elastic strain function $\mathbf{e}^e = [\mathbf{e}_1^e,\mathbf{e}_2^e,..,\mathbf{e}_n^e]$ and the plastic strain function $\mathbf{e}^p = [\mathbf{e}_1^p,\mathbf{e}_2^p,..,\mathbf{e}_n^p]$ satisfy the relations resulting from the balance of the internal and external work

$$\mathbf{e}_1^e(\mathbf{x},t)+\mathbf{e}_1^p(\mathbf{x},t) = \mathbf{r}^T(\mathbf{x},t)\boldsymbol{\varepsilon}(\mathbf{x},t)\mathbf{r}(\mathbf{x},t)$$

$$\mathbf{e}_i^e(\mathbf{x},t)+\mathbf{e}_i^p(\mathbf{x},t) = \mathbf{0} \quad \text{for } i=2,3,..,n \quad . \qquad (11.15)$$

9. The elastic strain function and the internal stress function satisfy the elastic deformation law

$$\mathbf{e}^e \in \partial\Psi(\mathbf{s}) \quad . \qquad (11.16)$$

10. The plastic strain rate function and the internal stress function satisfy the plastic flow law

$$\dot{\mathbf{e}}^p \in \partial\Phi(\mathbf{s}) \qquad (11.17)$$

provided that the free energy function $\psi^*(\mathbf{e}^e): T^* \to R^1$, the dissipation potential $\varphi(\mathbf{s}): T \to R^1$, the boundary displacement function $\mathbf{u}_o(\mathbf{x},t): \mathbf{B}_o^u \to R^3$ and the boundary force function $\mathbf{f}_o(\mathbf{x},t): \mathbf{B}_o^f \to R^3$ are prescribed □

Assuming that the continuous displacement function $\mathbf{u}$, the internal stress function $\mathbf{s}$ and the plastic strain rate function $\dot{\mathbf{e}}^p$ are basic unknowns we shall compress the formulation of the problem presented above.

Let us observe that the auxiliary unknowns can be simply determined by the basic ones. Namely, the total internal strain function $\mathbf{e} = [\mathbf{r}^T\boldsymbol{\varepsilon}\mathbf{r},\mathbf{0},..,\mathbf{0}]$ is determined by the displacement function $\mathbf{u}$ with the relation (11.5). The elastic strain function is determined by $\dot{\mathbf{e}}^p$ and $\mathbf{u}$ with the relations (11.5),(11.10) and $\mathbf{e}^e = \mathbf{e}-\mathbf{e}^p$.

For every displacement function $\mathbf{u}$ from the family C we define the set S_u of all internal stress functions $\mathbf{s}$ which are statically admissible (see section 9) and satisfy the initial (11.2) and the boundary (11.4) conditions.

We denote by K the set of all displacement functions $\mathbf{u}$ from the set C, which are integrable over the surface $\mathbf{B}$ of arbitrary region $\mathbf{V}$ from the family M and satisfy the initial (11.1) and the boundary (11.3) conditions.

The initial-boundary value problem may be now expressed as follows:

<u>Find the displacement function</u> $\mathbf{u} \in K$, <u>the internal stress function</u> $\mathbf{s} \in S_u \cap \mathrm{dom}\,\Phi \cap \mathrm{dom}\,\Psi$ <u>and the plastic strain rate function</u> $\dot{\mathbf{e}}^p \in \mathrm{dom}\,\Phi^*$ <u>such that</u>

(i) <u>The elastic strain function</u> $\mathbf{e}^e$, <u>determined by</u> $\mathbf{u}$ <u>and</u> $\dot{\mathbf{e}}^p$, <u>belongs to</u> dom Ψ^* <u>and satisfies the elastic deformation law</u>

$$\mathbf{e}^e \in \partial\Psi(\mathbf{s}) \quad \underline{\text{in the dual spaces}}\ P_e, P_e^* . \qquad (11.18)$$

(ii) *The plastic strain rate function satisfies the plastic flow law*

$$\dot{\mathbf{e}}^p \in \partial\Phi(\mathbf{s}) \quad \text{\underline{in the dual spaces} } P_p, P_p^* . \qquad (11.19)$$

12. MINIMUM PRINCIPLE

In order to formulate the minimum principle corresponding to the considered initial-boundary value problem we shall use the basic property of the sub-differential calculus. Namely, it follows directly from the definition of the sub-gradient and the polar function in the dual spaces (for example P_p and P_p^*) that the statements:

(i) Non-negative function $\Phi(\mathbf{s})-(\mathbf{s},\dot{\mathbf{e}}^p)+\Phi^*(\dot{\mathbf{e}}^p)$ defined for every $\mathbf{s}\in P_p$ and $\dot{\mathbf{e}}^p\in P_p^*$ is equal to zero.

(ii) The elements $\mathbf{s}\in P_p$ and $\dot{\mathbf{e}}^p\in P_p^*$ satisfy the relation $\mathbf{e}^p\in\partial\Phi(\mathbf{s})$.

are equivalent. Hence non-negative functional

$$\Phi(\mathbf{s})-(\mathbf{s},\dot{\mathbf{e}}^p)+\Phi^*(\dot{\mathbf{e}}^p)+\Psi(\mathbf{s})-(\mathbf{s},\mathbf{e}^e)+\Psi^*(\mathbf{e}^e) \qquad (12.1)$$

attains its absolute minimum equal to zero if and only if

$$\dot{\mathbf{e}}^p \in \partial\Phi(\mathbf{s}) \quad \text{and} \quad \mathbf{e}^e \in \partial\Psi(\mathbf{s}) \ . \qquad (12.2)$$

Taking into account that the function $\mathbf{e}^e$ is uniquely determined by $\mathbf{u}$ and $\dot{\mathbf{e}}^p$ (see section 11) we establish the minimum principle in the form:

The non-negative functional

$$F(\mathbf{u},\mathbf{s},\dot{\mathbf{e}}^p) = \Phi(\mathbf{s})-(\mathbf{s},\dot{\mathbf{e}}^p)+\Phi^*(\dot{\mathbf{e}}^p)+\Psi(\mathbf{s})-(\mathbf{s},\mathbf{e}^e)+\Psi^*(\mathbf{e}^e) \qquad (12.3)$$

defined for all $\mathbf{u}\in K$, $\mathbf{s}\in P_p\cap P_e$ *and* $\dot{\mathbf{e}}^p\in P_p^*$ *attains, under the constraint* $\mathbf{s}\in S_u$, *its absolute minimum equal to zero if and only if the functions* $\mathbf{u}$, $\mathbf{s}$, $\dot{\mathbf{e}}^p$ *represent the solution of the initial-boundary value problem.*

The form of the constraint implies a convenient sequence of the minimization process. Namely, for every $\mathbf{u}\in K$ one can calculate $G(\mathbf{u})$, which is defined as an absolute minimum of the function $F(\mathbf{u},\mathbf{s},\dot{\mathbf{e}}^p)$ in the corresponding domain of the stress and strain functions

$$G(\mathbf{u}) = \min\,[F(\mathbf{u},\mathbf{s},\dot{\mathbf{e}}^p):\ \mathbf{s}\in S_u\cap P_p\cap P_e\ ,\ \dot{\mathbf{e}}^p\in P_p^*]\ . \qquad (12.4)$$

In order to investigate the properties of the function $G(\mathbf{u})$ we introduce the set D_p of all functions $\mathbf{e}^p$ which are derived with (11.10) from plastically admissible strain rate function $\dot{\mathbf{e}}^p \in \mathrm{dom}\,\Phi^*$. It has been shown in section 9 that the set D_p is contained in $L^1(\mathbf{V}_o)$. Hence the set $\mathbf{e}-D_p$, where the total internal strain function $\mathbf{e}$ is determined by $\mathbf{u}$, represents the admissible domain for the elastic strain function $\mathbf{e}^e$. On the other hand the elastic strain function must be elastically admissible, i.e. $\mathbf{e}^e \in \mathrm{dom}\,\Psi^*$. Finally our requirement concerning $\mathbf{e}^e$ takes the form $\mathbf{e}^e\in D_e$, where the intersection

$$D_e = (\mathbf{e}-D_p)\cap \mathrm{dom}\,\Psi^* \qquad (12.5)$$

represents the set of all functions $\mathbf{e}^e$, which are kinematically, plastically and elastically admissible.

Similarly we introduce the domain D_s of all internal stress functions $\mathbf{s}$ which are statically, plastically and elastically admissible

$$D_s = S_u \cap \mathrm{dom}\,\Phi \cap \mathrm{dom}\,\Psi\ . \qquad (12.6)$$

It should be noted that $G(\mathbf{u})$ assumes the value $+\infty$ if the set D_e is empty or the set D_s is empty. In the remaining cases $G(\mathbf{u})$ is finite. According to the convention used in the work the set of all displacement functions $\mathbf{u}$ from the set K for which $G(\mathbf{u})<+\infty$ will be denoted by dom G.

Now the problem is reduced to the minimization of the function $G(\mathbf{u})$ in the set K. The function $G(\mathbf{u})$ attains an absolute minimum equal to zero if and only if the displacement function $\mathbf{u}$ represents the solution of the problem. If the absolute minimum of function $G(\mathbf{u})$ is positive then the solution of the initial-boundary value problem does not exist.

13. EXISTENCE AND UNIQUENESS OF SOLUTION

It follows from the formulation of the initial-boundary value problem that the prescribed input data must satisfy the following requirements:

(i) The boundary displacement function $\mathbf{u}_o(\mathbf{x},t)$ is integrable on $\mathbf{B}_o^u$ and there exists $\mathbf{u}\in K$ such that $\mathbf{u}(\mathbf{x},t)=\mathbf{u}_o(\mathbf{x},t)$ on $\mathbf{B}_o^u$. At the initial moment $\mathbf{u}_o(\mathbf{x},0)=\mathbf{0}$.

(ii) The boundary force function $\mathbf{f}_o(\mathbf{x},t)$ is integrable on $\mathbf{B}_o^f$ and $\mathbf{f}_o(\mathbf{x},0)=\mathbf{0}$.

(iii) The body force function $\mathbf{b}(\mathbf{x},t)$ is integrable in $\mathbf{V}_o$ and $\mathbf{b}(\mathbf{x},0)=\mathbf{0}$.

It is simple to show that the condition $\mathrm{dom}\, G \neq \emptyset$ is necessary but not sufficient for the existence of a solution. The solution exists if and only if the minimum value of the function $G(\mathbf{u})$ is equal to zero.

It should be noted that the statement "the solution does not exist" means that for the prescribed loading trajectory, represented by the functions $\mathbf{u}_o$, $\mathbf{f}_o$, $\mathbf{b}$, the requirements $\mathbf{1}\div\mathbf{10}$ listed in section 11 can not be satisfied simultaneously. Such situation may occur for a variety of reasons.

For example, solution does not exist if the elastic-perfectly plastic material is too weak to support the imposed loading. An other example may be given for purely elastic material, where the equilibrium of forces (in the frame of the quasi-static approach assumed throughout the work) can not be satisfied along the entire deformation path.

An illustration of the second example is given in Fig.15. Let the surface force $\mathbf{f}_o$ applied to the shell increase monotonously in time until the shell attains the final shape presented in the picture.

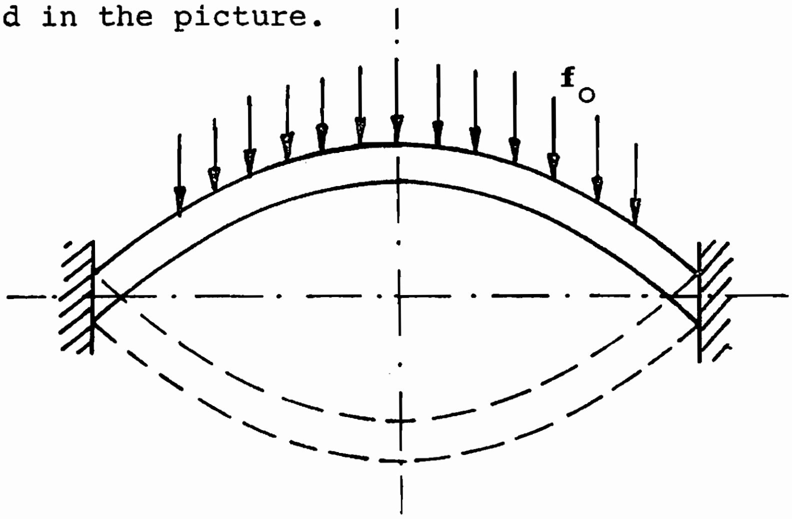

Fig.15. An example of the shell subjected to the surface force $\mathbf{f}_o$.

It is clear that the equilibrium of forces (11.7) and moments (11.8) along the entire deformation path can not be satisfied. Hence $G(\mathbf{u})$ can never become zero. It should be noted that the dynamic analysis, where the inertia terms are not neglected, would have ensured the existence of a solution. However, in the quasi-static approach applied here, the result $\min G(\mathbf{u}) > 0$ may be considered as a suitable criterion for the stability of the deformation process.

In general the solution of the initial-boundary value problem is not unique. There may exist a set of distinct displacement functions $\mathbf{u}$, for which the function $G(\mathbf{u})=0$. In other words one can obtain two or more distinct deformation paths for the body subjected to the prescribed external loading represented by the functions $\mathbf{b}$, $\mathbf{u}_o$ and $\mathbf{f}_o$. All deformation paths corresponding to the prescribed loading have the common segment, which includes the prescribed initial state. The last point of the common segment may be referred to as the first bifurcation point.

14. DISCONTINUOUS SOLUTION

The formulation of the initial-boundary value problem presented in section 11 admits only continuous deformation of the body. Namely, it is assumed there that the displacement function $\mathbf{u}(\mathbf{x},t)$ is continuous in the considered space-time region $\mathbf{V}_o$, i.e. it belongs to the set C.

Due to the above assumption the strain-displacement relation (11.5) for a region $\mathbf{V}$ of volume measure zero from the family M^o (see section 9) does not lead to a contradiction. Indeed, for every $\mathbf{u} \in C$ and $\mathbf{V} \in M^o$ the surface integral over the surface $\mathbf{B}$ of the region $\mathbf{V}$ vanishes

$$\int_{\mathbf{B}} \mathbf{u} \otimes \mathbf{n} \, dm_B = \mathbf{0} \qquad (14.1)$$

and the equation (11.5) holds true for arbitrary functions $\boldsymbol{\varepsilon}$ and $\mathbf{r}$, as $m(\mathbf{V}) = 0$.

In the present section we shall relax the restriction of continuity of the function $\mathbf{u}$ without violating the requirement of the compatibility of deformation. Namely, we introduce the set C_ε of all displacement functions $\mathbf{u}$ which are continuous in the region $\mathbf{V}_o$ with the exception of certain region $\mathbf{V}_\varepsilon$ of zero volume measure from the family M^o. It follows directly from this definition that $C \subset C_\varepsilon$.

We shall identify the region $\mathbf{V}_\varepsilon$ with the internal surface of the displacement discontinuity and we shall call it the slip surface. In our considerations the slip surface $\mathbf{V}_\varepsilon$ constitutes an additional unknown of the initial-boundary value problem.

In order to modify the formulation of the problem given in section 11 we introduce the family M_ε of regular regions, which consists of two sub-families M'_ε and M''_ε. Here the family M'_ε is the set of all regular regions $\mathbf{V}$ from M which do not intersect $\mathbf{V}_\varepsilon$

$$M'_\varepsilon = [\mathbf{V}: \mathbf{V} \in M \; , \; \mathbf{V} \cap \mathbf{V}_\varepsilon = \emptyset] \qquad (14.2)$$

and the family M''_ε is the set of all regular regions $\mathbf{V}$ of volume measure zero, which are contained in $\mathbf{V}_\varepsilon$

$$M''_\varepsilon = [\mathbf{V}:\ \mathbf{V}\in M^o\ ,\ \mathbf{V}\subset\mathbf{V}_\varepsilon]\ . \qquad (14.3)$$

Now the fields of strain and stress are defined on the family M_ε of the regular regions. For example the strain field $\mathbf{e}(\mathbf{V}):\ M_\varepsilon \to T^*$ is determined by (9.1) for $\mathbf{V}\in M'_\varepsilon$ and by

$$\mathbf{e}(\mathbf{V}) = \frac{1}{m_B(\mathbf{V})}\int_{\mathbf{V}} \mathbf{e}(\mathbf{x},t)\ dm_B \qquad \text{for } \mathbf{V}\in M''_\varepsilon\ . \qquad (14.4)$$

Similarly, following the construction given in section 10 we introduce the modified dual spaces P_e, P_e^* and P_p, P_p^*, determined by the modified global functions (14.8),(14.9), (14.10),(14.11) and provided with the bilinear form

$$(\mathbf{s},\mathbf{e}) = \int_{\mathbf{V}_o} \mathbf{s}(\mathbf{x},t)\cdot\mathbf{e}(\mathbf{x},t)\ dm + \int_{\mathbf{V}_\varepsilon} \mathbf{s}(\mathbf{x},t)\cdot\mathbf{e}(\mathbf{x},t)\ dm_B \quad (14.5)$$

Making use of the concept introduced above we formulate the initial-boundary value problem in which allowance for possible discontinuities in the displacements is made:

<u>Find the slip surface $\mathbf{V}_\varepsilon\in M^o$, the displacement function $\mathbf{u}(\mathbf{x},t):\ \mathbf{V}_o\to R^3$ from the set C_ε and the surface force function $\mathbf{f}(\mathbf{n},\mathbf{x},t):\ N\times\mathbf{V}_o\to R^3$ satisfying the initial conditions</u> (11.1), (11.2) <u>and the boundary conditions</u> (11.3),(11.4) <u>such that appropriately modified requirements **1÷10**, listed in section</u> 11, <u>are satisfied</u>.

The modifications consist in adding to the appropriate volume integrals the surface integrals resulting from the discontinuities and in replacing the family M of the regular regions by the family M_ε. Namely:

1. The equation (11.5) (requirement **1**) is replaced by

$$\int_{\mathbf{B}} \mathbf{u}\otimes\mathbf{n}\ dm_B = \int_{\mathbf{V}} (\exp\ \boldsymbol{\varepsilon}\ \mathbf{r} - \mathbf{1})\ dm + \int_{\mathbf{V}\cap\mathbf{V}_\varepsilon} (\exp\ \boldsymbol{\varepsilon}\ \mathbf{r} - \mathbf{1})\ dm_B . \ (14.6)$$

2. The equation (11.6) (requirement **2**) is replaced by

$$\int_{\mathbf{B}} \mathbf{f}\otimes(\mathbf{x}+\mathbf{u})\ dm_B + \int_{\mathbf{V}} \mathbf{b}\otimes(\mathbf{x}+u)\ dm = \int_{\mathbf{V}} \boldsymbol{\sigma}\ dm + \int_{\mathbf{V}\cap\mathbf{V}_\varepsilon} \boldsymbol{\sigma}\ dm_B \qquad (14.7)$$

3. The definition (11.9) (requirement **4**) is replaced by

$$\Phi^*(\dot{\mathbf{e}}^p) = \int_{\mathbf{V}_o} \varphi^*(\dot{\mathbf{e}}^p)\ dm + \int_{\mathbf{V}_\varepsilon} \varphi^*(\dot{\mathbf{e}}^p)\ dm_B\ . \qquad (14.8)$$

4. The definition (11.11) (requirement **5**) is replaced by

$$\Psi^*(\mathbf{e}^e) = \int_{\mathbf{V}_o} \psi^*(\mathbf{e}^e)\ dm + \int_{\mathbf{V}_\varepsilon} \psi^*(\mathbf{e}^e)\ dm_B\ . \qquad (14.9)$$

5. The definitions (11.12),(11.13) (requirement **6**) are replaced by

$$\Phi(\mathbf{s}) = \int_{\mathbf{V}_o} \varphi(\mathbf{s})\ dm + \int_{\mathbf{V}_\varepsilon} \varphi(\mathbf{s})\ dm_B \qquad (14.10)$$

$$\Psi(\mathbf{s}) = \int_{\mathbf{V}_o} \psi(\mathbf{s})\ dm + \int_{\mathbf{V}_\varepsilon} \psi(\mathbf{s})\ dm_B \qquad (14.11)$$

Let K_ε denote the set of all displacement functions $\mathbf{u}$ from the set C_ε, which are integrable over the surface $\mathbf{B}$ of arbitrary region $\mathbf{V}$ from the family M_ε and satisfy the initial condition (11.1) and the boundary condition (11.3).

Making use of formal similarity of the modified problem with the original one we can repeat the construction of the minimum principle, presented in section 12. As a result we obtain for every slip surface $\mathbf{V}_\varepsilon$ from the family M^o the non-negative function $G_1(\mathbf{V}_\varepsilon,\mathbf{u})$ defined on the set K_ε of the displacement functions $\mathbf{u}$. Minimizing the function G_1 in K_ε we obtain the non-negative function H

$$H(\mathbf{V}_\varepsilon) = \min[G_1(\mathbf{V}_\varepsilon,\mathbf{u}):\ \mathbf{u}\in K_\varepsilon] \qquad (14.12)$$

which is to be minimized in the family M^o of all possible internal slip surfaces $\mathbf{V}_\varepsilon$.

From the construction of the minimum principles it is clear that if there exists $\mathbf{u} \in K$ satisfying $G(\mathbf{u})=0$ then $H(\mathbf{V}_\varepsilon)=0$ for every $\mathbf{V}_\varepsilon \in M^0$. In other words a continuous solution of the problem may be considered as the particular discontinuous solution. This statement holds true for arbitrary $\mathbf{V}_\varepsilon \in M^0$ as the function $\mathbf{u}$ is continuous everywhere in $\mathbf{V}_0$.

On the other hand there exists a class of prescribed loadings for which the continuous solution does not exist, i.e. $\min[G(\mathbf{u}):\ \mathbf{u} \in K]>0$ while there exists the slip surface $\mathbf{V}_\varepsilon \in M^0$ and the discontinuous displacement function $\mathbf{u} \in K_\varepsilon$ which are the solution of the problem, i.e. $G_1(\mathbf{V}_\varepsilon,\mathbf{u})=0$.

Hence the result $\min[G(\mathbf{u}):\ \mathbf{u} \in K]>0$ excludes the existence of a continuous solution but does not exclude the existence of another types of solution.

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