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UNIFIED OPERATOR DESCRIPTION, NONLINEAR BUCKLING AND POST-BUCKLING ANALYSIS OF THIN ELASTIC SHELLS

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Summary: A nonlinear stability and post-buckling analysis based on energy considerations is performed for nonlinear elastic thin shells with moderate rotations. A unified operator description is introduced to formulate the nonlinear shell boundary value problem and to derive in a compact form the equations of critical equilibrium and the equations for a nonlinear post-buckling analysis with singular and multiple buckling modes. All equations are given with associated boundary conditions. The proposed operator description allows to outline the relations between the governing equations of the nonlinear shell boundary value problem and the buckling and post-buckling equations.

Übersicht: Ausgehend von energetischen Überlegungen wird eine nichtlineare Stabilitäts- und Nachbeulberechnung für dünne elastische Schalen mit moderaten Rotationen durchgeführt. Eine vereinfachende Operatorschreibweise des nichtlinearen Schalenrandwertproblems wird vorgeschlagen, die eine kompakte Herleitung der Gleichungen für kritisches Gleichgewicht sowie für eine nichtlineare Nachbeulberechnung mit einfachen und mehrfachen Beulformen ermöglicht. Alle Gleichungen werden unter Einschluß der zugehörigen Randbedingungen angegeben. Der vorgeschlagene Schalenoperator zeigt in einfacher Weise die Beziehungen zwischen den nichtlinearen Gleichungen des Schalenrandwertproblems und den Beul- sowie Nachbeulgleichungen.

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1. INTRODUCTION

The presented paper continues investigations, outlined in [1 - 5], on the subject of nonlinear elastic thin shells with moderate rotations. It is well-known that the so-called "moderate rotation shell theory" yields an adequate description for a very large range of nonlinear elastic shell problems of engineering interest. This theory, which is defined by the assumptions of small strains and moderate rotations, had been completed only recently by deriving the associated nonlinear boundary conditions [6 - 8]. The small strain restriction implies that "soft" materials are excluded. The governing equations of the nonlinear shell boundary value problem form a set of three nonlinear equilibrium equations together with four geometric and four nonlinear static boundary conditions. An equivalent description can be obtained as weak solution of associated variational principles [1 - 3].

The singularities of the governing equations can lead to bifurcation points with intersecting equilibrium paths and to points with snap-through phenomena associated with possible loss of stability. A significant subject of a nonlinear shell analysis is therefore the determination of critical points with bifurcation or snap-through buckling and the calculation of the load carrying capacity of the structure. Many investigations of shell stability are based on a linear approach by neglecting the deformations that occur prior to buckling. For some structures this procedure yields satisfactory results in predicting the critical loads, for many other problems the prebuckling deformations cannot be neglected and a linear approach leads to severe errors.

A more realistic approximation procedure than a linear stability analysis of "perfect" structures had been obtained by the introduction of "initial imperfections" [9 - 11]. Their influence on the stability behaviour is similar to the contribution of the nonlinear prebuckling deformations. For the moderate rotation shell theory the stability equations with associated boundary conditions had been derived by the author for linear and nonlinear prebuckling [4,5].

Using the total potential energy Koiter [9-11] developed an asymptotic technique to approximate the initial post-buckling behaviour and the imperfection sensitivity of structures. This method is applicable to structures with bifurcation type buckling. Sewell [12,13], Thompson [14], Thompson and Hunt [15] applied the static perturbation method to buckling problems of discrete systems and Budiansky [16] to the buckling of continuous systems. This perturbation technique is an appropriate procedure for bifurcation as well as snap-through type buckling of elastic structures and had been applied to the post-buckling analysis of frames [17 - 22], plates [23] and spherical shells [24]. Finite element procedures for the buckling analysis can be found in [25 - 28].

In this paper the nonlinear stability and post-buckling analysis of elastic shell structures is considered in the frame of the nonlinear shell theory with moderate rotations. Variational approaches as well as full sets of Lagrangian shell equations will be given to determine the initial nonlinear fundamental equilibrium path, the critical points of bifurcation or snap-through buckling and also the post-buckling equilibrium paths. To obtain a unified description and to clarify the relations between the prebuckling, buckling and post-buckling analysis a nonlinear shell operator is introduced, representing the field equations as well as the boundary conditions of the shell. Then the stability equations for nonlinear prebuckling can be obtained as differential of this shell operator. Using the static perturbation technique a systematic approach to the study of post-buckling paths with singular as well as multiple buckling modes is considered.

2. BASIC SHELL RELATIONS

In this section we start with the basic shell relations, which are outlined in more detail in [6 - 8]. All considerations are based on the physical assumptions that the shell is elastic, homogeneous and isotropic and of constant thickness h with $(h/R) \ll 1$, where R is the smallest principal radius of curvature of the undeformed shell middle surface. All shell deformations are assumed to be such that $(h/L)^2 \ll 1$, where L is the smallest wave length of deformation pattern of the shell middle surface. Furthermore we presume that the strains are small everywhere in the shell, while the strain-displacement relations are nonlinear.

To obtain a Lagrangean description of the nonlinear shell theory all shell deformations will be referred to the undeformed shell middle surface $\mathcal{M}$. Let $\boldsymbol{r}(\theta^\alpha)$, $\alpha = 1,2$ be the position vector from a fixed Cartesian coordinate system to an arbitrary point of the reference shell middle surface as a function of a set of curvilinear Gaussian coordinates θ^α, $\alpha = 1,2$. With θ^α we associate covariant base vectors $\boldsymbol{a}_\alpha = \boldsymbol{r}_{,\alpha}$ and a unit normal vector $\boldsymbol{n}$. The covariant components of the surface metric tensor $a_{\alpha\beta}$ and of the surface curvature tensor are given by the scalar products $a_{\alpha\beta} = \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta$ and $b_{\alpha\beta} = \boldsymbol{a}_{\alpha,\beta} \cdot \boldsymbol{n}$, where subscripts preceded by a comma indicate partial differentiation with respect to the corresponding coordinate direction. Using Einstein's summation convention the contravariant components $a^{\alpha\beta}$ of the metric tensor are defined by the relations $a^{\alpha\lambda} a_{\lambda\beta} = \delta^\alpha_\beta$, where δ^α_β is the Kronecker symbol. Here and in all following formulas Greek indices take the values 1,2.

The deformation of the shell middle surface from the undeformed reference configuration $\mathcal{M}$ into the deformed configuration $\bar{\mathcal{M}}$ can be described by the displacement field

$$\boldsymbol{u} = \bar{\boldsymbol{r}} - \boldsymbol{r} = u^{\alpha}\boldsymbol{a}_{\alpha} + u_3\boldsymbol{n}\,, \tag{2.1}$$

where u^{α} are the contravariant displacement components tangent to the middle surface and u_3 is the displacement component normal to the middle surface.

With the deformed shell middle surface $\bar{M}$ are associated corresponding covariant base vectors $\bar{\boldsymbol{a}}_{\alpha}$ and a unit normal vector $\bar{\boldsymbol{n}}$ defining the surface metric tensor $\bar{a}_{\alpha\beta}$ and the surface curvature tensor $\bar{b}_{\alpha\beta}$ of the deformed shell middle surface.

Assuming that the state of stress is approximately plane and parallel to the middle surface or equivalently that the Kirchhoff-Love hypothesis is valid, the shell deformation can be described by the middle surface strain tensor $\gamma_{\alpha\beta}$ and the change of curvature tensor $\kappa_{\alpha\beta}$

$$\gamma_{\alpha\beta} = \frac{1}{2}\,(\bar{a}_{\alpha\beta} - a_{\alpha\beta}) \quad , \quad \kappa_{\alpha\beta} = -(\bar{b}_{\alpha\beta} - b_{\alpha\beta}) \ . \tag{2.2}$$

Let $\theta_{\alpha\beta}$ be the linearized middle surface strain tensor and φ_i, $i = 1,2,3$ the linearized rotations defined by

$$\begin{aligned} &\theta_{\alpha\beta} = \frac{1}{2}\,(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta}u_3 \qquad && \omega_{\alpha\beta} = \frac{1}{2}\,(u_{\beta|\alpha} - u_{\alpha|\beta}) = \epsilon_{\alpha\beta}\varphi_3 \\ &\varphi_{\alpha} = u_{3,\alpha} + b^{\lambda}_{\alpha}u_{\lambda} && \varphi_3 = \frac{1}{2}\,\epsilon^{\alpha\beta}u_{\beta|\alpha} \end{aligned} \tag{2.3}$$

with the contravariant surface permutation tensor

$$\epsilon^{\alpha\beta} = \frac{1}{\sqrt{a}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and $a = \det(a_{\alpha\beta})$. In the frame of a nonlinear shell theory with moderate rotations the strain measures (2.2) can be approximated by [6]:

$$\gamma_{\alpha\beta} = \theta_{\alpha\beta} + \underline{\tfrac{1}{2}\varphi_\alpha\varphi_\beta} + \underline{\tfrac{1}{2} a_{\alpha\beta}\varphi_3^2} - \underline{\underline{\tfrac{1}{2}(\theta^\lambda_\alpha\omega_{\lambda\beta} + \theta^\lambda_\beta\omega_{\lambda\alpha})}} \quad ,$$

$$\kappa_{\alpha\beta} = -\frac{1}{2}\left[\varphi_{\alpha|\beta} + \varphi_{\beta|\alpha} + b^\lambda_\alpha(\underset{\cdots}{\theta_{\lambda\beta}} - \underset{-\cdot-}{\omega_{\lambda\beta}}) + b^\lambda_\beta(\underset{\cdots}{\theta_{\lambda\alpha}} - \underset{-\cdot-}{\omega_{\lambda\alpha}})\right] \tag{2.4}$$

where a vertical stroke preceding a subscript indicates covariant differentiation with respect to the corresponding coordinate direction.

With the modified tensor of elastic moduli

$$H^{\alpha\beta\lambda\mu} = \frac{Eh}{2(1+\nu)}\left(a^{\alpha\lambda}a^{\beta\mu} + a^{\alpha\mu}a^{\beta\lambda} + \frac{2\nu}{1-\nu}a^{\alpha\beta}a^{\lambda\mu}\right) \tag{2.5}$$

the stress resultant tensor $N^{\alpha\beta}$ and the stress couple tensor $M^{\alpha\beta}$ are given by the constitutive relations:

$$N^{\alpha\beta} = H^{\alpha\beta\lambda\mu}\gamma_{\lambda\mu} = \frac{Eh}{1-\nu^2}\left[(1-\nu)\gamma^{\alpha\beta} + \nu a^{\alpha\beta}\gamma^\lambda_\lambda\right]$$

$$M^{\alpha\beta} = \frac{h^2}{12}H^{\alpha\beta\lambda\mu}\kappa_{\lambda\mu} = \frac{Eh^3}{12(1-\nu^2)}\left[(1-\nu)\kappa^{\alpha\beta} + \nu a^{\alpha\beta}\kappa^\lambda_\lambda\right] \quad , \tag{2.6}$$

where E is Young's modulus and ν Poisson's ratio.

Assuming that distributed surface loads per unit area of the undeformed middle surface are given:

$$\mathbf{p} = p^\alpha \mathbf{a}_\alpha + p_3\mathbf{n} \tag{2.7}$$

the nonlinear equilibrium equations for the moderate rotation shell theory are obtained in the following form [3,6,7]

$$\Big[N^{\alpha\beta} - \underset{\cdots\cdots\cdots}{\tfrac{1}{2}(b^\alpha_\lambda M^{\lambda\beta} + b^\beta_\lambda M^{\lambda\alpha})} - \underset{-\cdot-\cdot-\cdot-}{\tfrac{1}{2}(b^\alpha_\lambda M^{\lambda\beta} - b^\beta_\lambda M^{\lambda\alpha})} - \underline{\tfrac{1}{2}\omega^{\alpha\beta}N^\lambda_\lambda} - \underline{\underline{\tfrac{1}{2}(\omega^{\alpha\lambda}N^\beta_\lambda + \omega^{\beta\lambda}N^\alpha_\lambda)}} +$$

$$+ \underline{\underline{\tfrac{1}{2}(\theta^{\alpha\lambda}N^\beta_\lambda - \theta^{\beta\lambda}N^\alpha_\lambda)}}\Big]\Big|_\beta - b^\alpha_\beta(M^{\lambda\beta}|_\lambda + \underline{\varphi_\lambda N^{\lambda\beta}}) + p^\alpha = 0 \tag{2.8}$$

$$(M^{\alpha\beta}|_\alpha + \underline{\varphi_\alpha N^{\alpha\beta}})|_\beta + b_{\alpha\beta}[N^{\alpha\beta} - \underset{\cdots}{b^\alpha_\lambda M^{\lambda\beta}} - \underline{\underline{\omega^{\alpha\lambda}N^\beta_\lambda}}] + p_3 = 0$$

Expressing $N^{\alpha\beta}$ and $M^{\alpha\beta}$ by the strain measures $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ and introducing then the strain-displacement relations (2.3) - (2.4) the equilibrium equations (2.8) yield a set of three nonlinear partial differential equations in the displacement components u_α, u_3 of the shell middle surface.

To complete the set of governing equations by the boundary conditions we introduce an orthonormal vector system $\underset{\sim}{\nu}$, t, n at the line C bounding the undeformed shell middle surface M:

$$t = \frac{dr}{ds} = t^\alpha a_\alpha \;;\quad \underset{\sim}{\nu} = t \times n = \nu^\alpha a_\alpha = \epsilon_{\beta\alpha} t^\alpha a^\beta \;;\quad n = \frac{1}{2}\epsilon^{\alpha\beta} a_\alpha \times a_\beta \,, \tag{2.9}$$

where s is the length parameter of C. A cross indicates the usual vector product.

With the physical components $u_\nu = u^\alpha \nu_\alpha$, $u_t = u^\alpha t_\alpha$ and u_3 the displacement vector u at the boundary can be expressed by

$$u = u_\nu \underset{\sim}{\nu} + u_t t + u_3 n \,. \tag{2.10}$$

As a fourth independent geometric quantity at the boundary the rotation about the tangent to the boundary line $\beta_\nu = -\varphi_\alpha \nu^\alpha$ will be used.

To describe the static boundary conditions the resultant stress vector T_ν and the resultant stress couple vector M_ν at the boundary are introduced, defined by

$$\begin{aligned} T_\nu &= T^\beta \nu_\beta = T_{\nu\nu} \underset{\sim}{\nu} + T_{t\nu} t + T_{n\nu} n \\ M_\nu &= M^\beta \nu_\beta = -M_{t\nu} \underset{\sim}{\nu} + M_{\nu\nu} t \end{aligned} \tag{2.11}$$

with:

$$\begin{aligned} T^\beta = [N^{\alpha\beta} &- \underset{\cdots\cdots\cdots}{\frac{1}{2}(b^\alpha_\lambda M^{\lambda\beta} + b^\beta_\lambda M^{\lambda\alpha})} - \underset{-\cdot-\cdot-\cdot-}{\frac{1}{2}(b^\alpha_\lambda M^{\lambda\beta} - b^\beta_\lambda M^{\lambda\alpha})} - \underline{\frac{1}{2}\omega^{\alpha\beta} N^\lambda_\lambda} - \underline{\underline{\frac{1}{2}(\omega^{\alpha\lambda} N^\beta_\lambda + \omega^{\beta\lambda} N^\alpha_\lambda)}} + \\ &+ \underline{\underline{\frac{1}{2}(\theta^{\alpha\lambda} N^\beta_\lambda - \theta^{\beta\lambda} N^\alpha_\lambda)}}]a_\alpha + (M^{\alpha\beta}|_\alpha + \underset{----}{\varphi_\alpha N^{\alpha\beta}})n \end{aligned} \tag{2.12}$$

$$M^\beta = \epsilon_{\alpha\lambda} M^{\alpha\beta} a^\lambda \,.$$

In (2.11) $T_{\nu\nu}$, $T_{t\nu}$, $T_{n\nu}$ and $M_{t\nu} = M^{\alpha\beta} t_\alpha \nu_\beta$, $M_{\nu\nu} = M^{\alpha\beta} \nu_\alpha \nu_\beta$ are physical quantities of the resultant stress and stress couple vector at the boundary.

Let us assume that on a part C_u of the boundary line C geometrical quantities and on a part C_f statical quantities are prescribed. If geometrical and statical quantities are given on the same part of the boundary line, they must be complementary to each other [1]. Then we have the geometrical boundary conditions on C_u

$$u_\nu = u^*_\nu \; ; \; u_t = u^*_t \; ; \; u_3 = u^*_3 \; ; \; \beta_\nu = \beta^*_\nu \; ; \; u_{3j} = u^*_{3j} \; , \; j = 1,\ldots,N_u \tag{2.13}$$

and the statical boundary conditions on C_f [3,6,7]:

$$\begin{aligned}
&\nu_\alpha \nu_\beta [N^{\alpha\beta} - \underline{b^\alpha_\lambda M^{\lambda\beta}} - \underline{\underline{\omega^{\alpha\lambda} N^\beta_\lambda}}] - b_{\nu t} M_{t\nu} = T^*_{\nu\nu} - b_{\nu t} M^*_{t\nu} = P^*_\nu \\
&t_\alpha \nu_\beta [N^{\alpha\beta} - \underline{\tfrac{1}{2}(b^\alpha_\lambda M^{\lambda\beta} + b^\beta_\lambda M^{\lambda\alpha})} - \underline{\tfrac{1}{2}(b^\alpha_\lambda M^{\lambda\beta} - b^\beta_\lambda M^{\lambda\alpha})} - \underline{\tfrac{1}{2}\omega^{\alpha\beta} N^\lambda_\lambda} - \underline{\underline{\tfrac{1}{2}(\omega^{\alpha\lambda} N^\beta_\lambda + \omega^{\beta\lambda} N^\alpha_\lambda)}} + \\
&\qquad + \underline{\underline{\tfrac{1}{2}(\theta^{\alpha\lambda} N^\beta_\lambda - \theta^{\beta\lambda} N^\alpha_\lambda)}}] - b_{tt} M_{t\nu} = T^*_{t\nu} - b_{tt} M^*_{t\nu} = P^*_t \\
&\nu_\beta (M^{\alpha\beta}|_\alpha + \underline{\varphi_\alpha N^{\alpha\beta}}) + \frac{d}{ds} M_{t\nu} = T^*_{n\nu} + \frac{d}{ds} M^*_{t\nu} = P^*_3 \\
&\nu_\alpha \nu_\beta M^{\alpha\beta} = M^*_{\nu\nu} \\
&M_{t\nu}(s_{fj} + 0) - M_{t\nu}(s_{fj} - 0) = M^*_{t\nu}(s_{fj} + 0) - M^*_{t\nu}(s_{fj} - 0) = F^*_j \; , \; j = 1,\ldots,N_f \; ,
\end{aligned} \tag{2.14}$$

where $b_{\nu t} = \nu^\alpha \nu^\beta b_{\alpha\beta}$ is the geometric torsion and $b_{tt} = t^\alpha t^\beta b_{\alpha\beta}$ the normal curvature of C. Given values are indicated by an asterisk. If on the part C_u discrete corner points are located at $s = s_{uj}$, $j = 1,\ldots,N_u$ and if on the part C_f discrete corner points are located at $s = s_{fj}$, $j = 1,\ldots,N_f$, then $(2.13)_5$ and $(2.14)_5$ are the geometrical and statical corner conditions. For all further considerations it will be presumed that external loads and boundary forces are of dead-load type.

The equilibrium equations (2.8) expressed as functions of the middle surface displacement field u form together with the geometric boundary conditions (2.13) and the static boundary conditions (2.14) the full set of nonlinear partial differential equations describing the nonlinear boundary value problem of elastic shells with moderate rotations. The solution of this boundary value problem yields the displacement field u depending nonlinearly on the given external loads.

3. SIMPLIFIED NONLINEAR SHELL THEORIES

As outlined in more details in [3] the governing equations of the moderate rotation shell theory include the equations of many well-known nonlinear and linear theories as special cases, which are obtained by omitting underlined quantities in the equations of section 2:

Omission of terms marked by (additional restriction)	yields the theory
════	Leonard [30]
════ • • • • •	Sanders [31]: "moderately small rotations"
	Koiter [32]: "small finite deflections"
════ • • • • • — · — · —	Pietraszkiewicz [1]
════ —— • • • • • — · — · —	Sanders [31]: "moderately small rotations about tangents and small rotations about normals to the middle surface" Koiter [32]
════ —— • • • • • — · — · — ($\varphi_\alpha = u_{3,\alpha}$)	Donnell-Marguerre-Vlasov [33-35]: "quasi-shallow shells" ($\gamma_{\alpha\beta} = \theta_{\alpha\beta} + \frac{1}{2} u_{3,\alpha} u_{3,\beta}$; $\kappa_{\alpha\beta} = -u_{3\|\alpha\beta}$)

═══ ─── ••••• —·—·— ($b_{\alpha\beta}$ = 0)	von Kármán [36] : nonlinear plate theory
═══ ─── – – – –	Naghdi [37] : linear shell theory
═══ ─── – – – – – •••••	Budiansky and Sanders [38]: linear shell theory

Table 1

By omission of marked terms according to table 1 all results of the following sections are valid also for these reduced shell theories.

4. APPROPRIATE OPERATOR FORM OF THE LINEAR SHELL THEORY

In order to derive the equations defining the critical points of bifurcation or snap-through buckling and to determine the post-buckling equilibrium paths of shell structures in a unified and compact form we introduce an appropriate operator description of the shell theory. In this section we start with the linear shell theory and define linear shell operators with associated bilinear functionals which allow to investigate important properties of the shell operator as symmetry, positivity or positive definiteness. They are of special interest for the construction of numerical approximation procedures like finite element methods. It should be mentioned here that the structure of consistent linear shell theories have been considered also in [39,40].

From the relations of section 2 the linear shell equations are obtained by neglecting all nonlinear terms. Assuming that the displacements $u \in H_u$ are elements of a Hilbert-space H_u and that the middle surface forces $p \in H_p$ are elements of a Hilbert-space H_p, we can put them into duality by a bilinear form

$$(u,p) = \iint_M (u_\alpha p^\alpha + u_3 p_3)\,dS \quad , \tag{4.1}$$

representing the work of the middle surface forces p applied to the displacement field u.

Furthermore we define elements $\varepsilon \in H_\varepsilon$ representing the state of strains and $\sigma \in H_\sigma$ representing the state of stresses of the shell:

$$\varepsilon = \begin{pmatrix} \theta_{11} \\ \theta_{22} \\ \theta_{12} + \theta_{21} \\ \kappa_{11} \\ \kappa_{22} \\ \kappa_{12} + \kappa_{21} \end{pmatrix} \in H_\varepsilon \quad , \qquad \sigma = \begin{pmatrix} N_\ell^{11} \\ N_\ell^{22} \\ N_\ell^{12} \\ M^{11} \\ M^{22} \\ M^{12} \end{pmatrix} \in H_\sigma \; . \tag{4.2}$$

The spaces H_ε and H_σ can be put into duality by the bilinear form:

$$\langle \varepsilon , \sigma \rangle = \iint\limits_{M} (\theta_{\alpha\beta} N_\ell^{\alpha\beta} + \kappa_{\alpha\beta} M^{\alpha\beta}) \, dS \tag{4.3}$$

with the linear stress resultant tensor:

$$N_\ell^{\alpha\beta} = H^{\alpha\beta\lambda\mu} \theta_{\lambda\mu} \quad . \tag{4.4}$$

The bilinear form (4.3) is the interaction energy of two linear shell deformations described by ε and σ.

We define now an operator T mapping elements $u \in \mathcal{D}_T \subset H_u$ into H_ε by

$$Tu = \begin{pmatrix} \theta_{11}(u) \\ \theta_{22}(u) \\ \theta_{12}(u) + \theta_{21}(u) \\ \kappa_{11}(u) \\ \kappa_{22}(u) \\ \kappa_{12}(u) + \kappa_{21}(u) \end{pmatrix} \quad , \tag{4.5}$$

where in the subset $\mathcal{D}_T \subset \mathcal{H}_u$ sufficient differentiability conditions have to be satisfied and rigid body motions are excluded.

The constitutive equations of the linear shell theory can be expressed by using an operator $\mathsf{H}\colon \mathcal{H}_\varepsilon \rightarrow \mathcal{H}_\sigma$

$$\mathsf{H}\varepsilon = \begin{pmatrix} N_\ell^{11}(\varepsilon) \\ N_\ell^{22}(\varepsilon) \\ N_\ell^{12}(\varepsilon) \\ M^{11}(\varepsilon) \\ M^{22}(\varepsilon) \\ M^{12}(\varepsilon) \end{pmatrix} \tag{4.6}$$

where $N_\ell^{\alpha\beta}(\theta_{\lambda\mu})$ and $M^{\alpha\beta}(\kappa_{\lambda\mu})$ are given by (4.4) and $(2.6)_2$:

Furthermore an operator $\mathsf{T}^*\colon \mathcal{H}_\sigma \rightarrow \mathcal{H}_p$ is introduced by

$$\mathsf{T}^*\sigma = -\begin{pmatrix} [N_\ell^{\alpha\beta} - \frac{1}{2}(b_\lambda^\alpha M^{\lambda\beta} + b_\lambda^\beta M^{\lambda\alpha}) - \frac{1}{2}(b_\lambda^\alpha M^{\lambda\beta} - b_\lambda^\beta M^{\lambda\alpha})]|_\beta - b_\beta^\alpha M^{\lambda\beta}|_\lambda \\ \\ M^{\alpha\beta}|_{\alpha\beta} + b_{\alpha\beta}(N_\ell^{\alpha\beta} - b_\lambda^\alpha M^{\lambda\beta}) \end{pmatrix} \in \mathcal{H}_p \tag{4.7}$$

which will be used to formulate the linear shell equilibrium equations in operator form. It is shown in equation (4.16) that the operator T^* is formally adjoint to the operator T.

The given definitions of linear shell operators and bilinear forms are represented graphically in the following scheme (fig. 1):

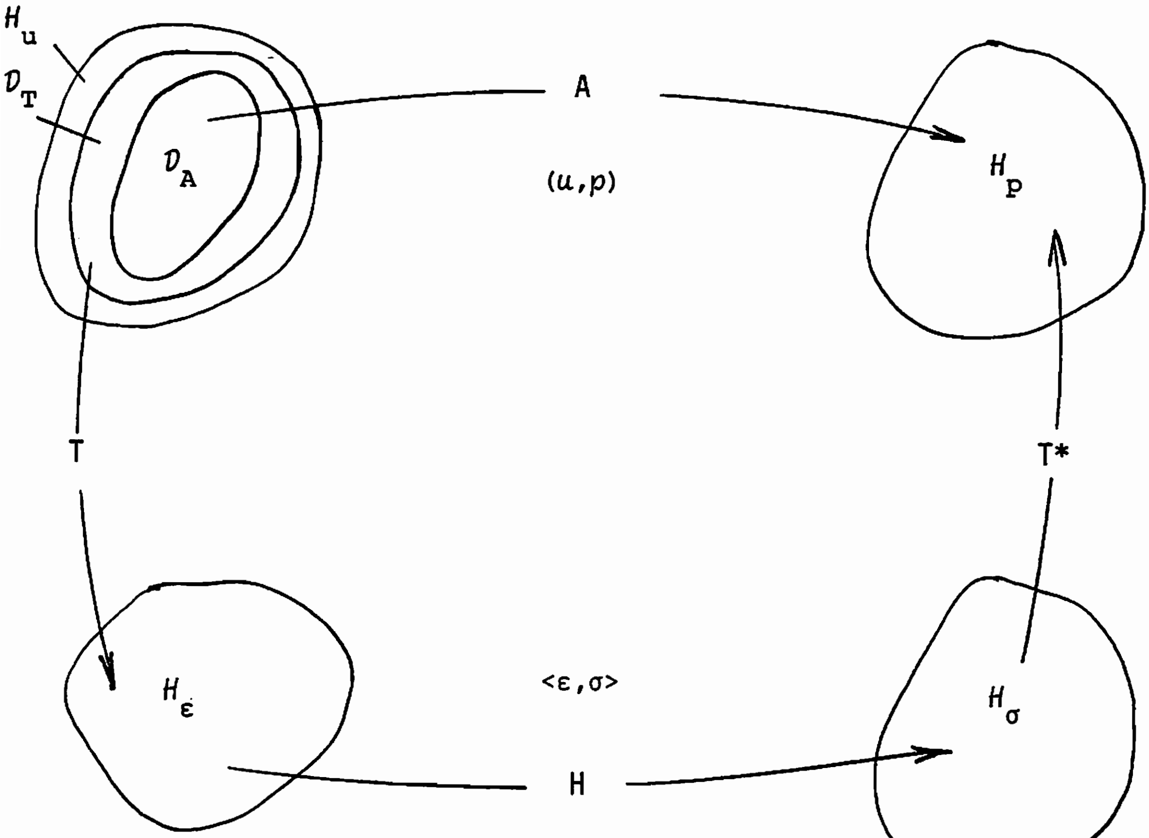

Fig. 1

According to fig. 1 we can define an operator A with the domain $\mathcal{D}_A \subset \mathcal{D}_T$ and with the range H_p by

$$Au = T^*HTu = -\left\{\begin{matrix} [N_\ell^{\alpha\beta}(u) - \underline{\frac{1}{2}(b^\alpha_\lambda M^{\lambda\beta}(u) + b^\beta_\lambda M^{\lambda\alpha}(u))} - \underline{\frac{1}{2}(b^\alpha_\lambda M^{\lambda\beta}(u) - b^\beta_\lambda M^{\lambda\alpha}(u))}]\big|_\beta - b^\alpha_\beta M^{\lambda\beta}(u)\big|_\lambda \\ \\ M^{\alpha\beta}(u)\big|_{\alpha\beta} + b_{\alpha\beta}(N_\ell^{\alpha\beta}(u) - \underline{b^\alpha_\lambda M^{\lambda\beta}(u)}) \end{matrix}\right\} \in H_p \tag{4.8}$$

which allows to formulate the linear equilibrium equations of the shell theory in the operator form:

$$Au = p \tag{4.9}$$

In a next step we have to introduce appropriate boundary operators. In the frame of the considered linear and nonlinear, respectively, shell theory we define a geometric boundary vector u_b representing the three physical components of the displacements vector u and the rotation β_ν around the boundary line C together with the normal displacement u_{3j} at the discrete corner points $j=1,\ldots$. Correspondingly we define a static boundary vector P_b collecting the three components P_ν, P_t, P_3, the boundary stress couple $M_{\nu\nu}$ and the corner forces F_j at the discrete corner points:

$$u_b = \begin{bmatrix} u_\nu \\ u_t \\ u_3 \\ \beta_\nu \\ u_{3j},\ j=1,\ldots \end{bmatrix}, \qquad P_b = \begin{bmatrix} P_\nu \\ P_t \\ P_3 \\ M_{\nu\nu} \\ F_j,\ j=1,\ldots \end{bmatrix} \tag{4.10}$$

We put these geometric and static boundary vectors into duality by the line integral:

$$[u_b, P_b] = \int_C (u_\nu P_\nu + u_t P_t + u_3 P_3 + \beta_\nu M_{\nu\nu})\,ds + \sum_{j=1}^{N} u_{3j} F_j \tag{4.11}$$

With (4.10) we define the following boundary operators:

$$\pi u = \begin{bmatrix} \nu_\alpha u^\alpha \\ t_\alpha u^\alpha \\ u_3 \\ -\nu^\alpha (u_{3,\alpha} + b^\lambda_\alpha u_\lambda) \\ u_{3j},\ j=1,\ldots \end{bmatrix}, \quad \rho\sigma = \begin{bmatrix} \nu_\alpha \nu_\beta (N^{\alpha\beta}_\ell - \underset{\cdots\cdots}{b^\alpha_\lambda M^{\lambda\beta}}) - b_{\nu t} M_{t\nu} \\ t_\alpha \nu_\beta [N^{\alpha\beta}_\ell - \underset{\cdots\cdots\cdots\cdots\cdots}{\tfrac{1}{2}(b^\alpha_\lambda M^{\lambda\beta} + b^\beta_\lambda M^{\lambda\alpha})} - \underset{-\cdot-\cdot-\cdot-\cdot-}{\tfrac{1}{2}(b^\alpha_\lambda M^{\lambda\beta} - b^\beta_\lambda M^{\lambda\alpha})}] - b_{tt} M_{t\nu} \\ \nu_\beta M^{\alpha\beta}|_\alpha + \frac{d}{ds} M_{t\nu} \\ \nu_\alpha \nu_\beta M^{\alpha\beta} \\ M_{t\nu}(s_{fj}+0) - M_{t\nu}(s_{fj}-0),\ j=1,\ldots,N_f \end{bmatrix} \tag{4.12}$$

Corresponding to the operator A of the linear shell equilibrium equations we obtain a boundary operator a

$$
au = \rho HTu = \begin{Bmatrix}
\nu_\alpha \nu_\beta (N_\ell^{\alpha\beta}(u) - \underline{b_\lambda^\alpha M^{\lambda\beta}}(u)) - b_{\nu t} M_{t\nu}(u) \\
t_\alpha \nu_\beta [N_\ell^{\alpha\beta}(u) - \underline{\tfrac{1}{2}(b_\lambda^\alpha M^{\lambda\beta}(u) + b_\lambda^\beta M^{\lambda\alpha}(u))} - \underline{\tfrac{1}{2}(b_\lambda^\alpha M^{\lambda\beta}(u) - b_\lambda^\beta M^{\lambda\alpha}(u))}] - b_{tt} M_{t\nu}(u) \\
\nu_\beta M^{\alpha\beta}|_\alpha(u) + \frac{d}{ds} M_{t\nu}(u) \\
\nu_\alpha \nu_\beta M^{\alpha\beta}(u) \\
M_{t\nu}(u)(s_{fj}+0) - M_{t\nu}(u)(s_{fj}-0) \;, \; j = 1,\dots,N_f
\end{Bmatrix} , \quad (4.13)
$$

which will allow to present the static boundary conditions in operator form.

The bilinear form (4.3) defines an energy product for linear shell deformations. For a strain field $\varepsilon(u)$ and a stress field $\sigma(v)$ (4.3) yields the interaction energy of the two shell deformations u and v:

$$
\langle \varepsilon(u) \;,\; \sigma(v) \rangle = \iint_M [\theta_{\alpha\beta}(u) N_\ell^{\alpha\beta}(v) + \kappa_{\alpha\beta}(u) M^{\alpha\beta}(v)] ds =
$$

$$
= \iint_M H^{\alpha\beta\lambda\mu} [\theta_{\alpha\beta}(u)\theta_{\lambda\mu}(v) + \frac{h^2}{12} \kappa_{\alpha\beta}(u)\kappa_{\lambda\mu}(v)] ds \quad , \quad (4.14)
$$

which, under certain restrictions, allows to define an energy norm

$$
|||\, u \,|||^2 = \langle \varepsilon(u) \;,\; \sigma(u) \rangle = \iint_M H^{\alpha\beta\lambda\mu} [\theta_{\alpha\beta}(u)\theta_{\lambda\mu}(u) + \frac{h^2}{12} \kappa_{\alpha\beta}(u)\kappa_{\lambda\mu}(u)] ds =
$$

$$
= \langle HTu, Tu \rangle > 0 \quad \text{for} \quad u \neq 0 \quad , \quad (4.15)
$$

because the operator H is positive definite for well-chosen elastic coefficients of (2.5).

With (4.1) and (4.14) it can be shown by partial integration that the operator T^*, given by (4.7) is formally adjoined to the operator T according to (4.5):

$$
\langle Tu, \sigma \rangle = (u, T^*\sigma) + [u_b, \rho\sigma] . \quad (4.16)
$$

It can be proved also that the operator A of the linear shell boundary value problem is symmetric in a formal sense:

$$(Au,v) = (u,Av) - [\rho HTu,v_b] + [\rho HTv,u_b] \quad . \tag{4.17}$$

With the operators, introduced in this section, we formulate the linear shell boundary value problem in the following compact form:

$$\begin{aligned} Au &= p \quad &&\text{in } M \\ \pi u &= u_b^* \quad &&\text{on } C_u \quad , \\ au &= P_b^* \quad &&\text{on } C_f \end{aligned} \tag{4.18}$$

where p are the given middle surface forces, u_b^* the given geometrical boundary datas on C_u and P_b^* the given statical boundary datas on C_f. As solution of the boundary value problem (4.18) the displacement field u of the shell middle surface has to be determined.

5. NONLINEAR SHELL BOUNDARY VALUE PROBLEM IN OPERATOR DESCRIPTION

For the nonlinear elastic shell theory with moderate rotations the middle surface strains $\gamma_{\alpha\beta}$ depend nonlinearly on the displacement field u. For further considerations we split the stress resultants $N^{\alpha\beta}$ in a linear part $N_\ell^{\alpha\beta}$ and a nonlinear part $N_n^{\alpha\beta}$

$$N^{\alpha\beta} = N_\ell^{\alpha\beta} + N_n^{\alpha\beta} \tag{5.1}$$

with

$$\begin{aligned} N_\ell^{\alpha\beta} &= H^{\alpha\beta\lambda\mu}\theta_{\lambda\mu} \\ N_n^{\alpha\beta} &= H^{\alpha\beta\lambda\mu}\frac{1}{2}[\underline{\varphi_\lambda\varphi_\mu} + \underline{a_{\lambda\mu}\varphi_3^2} - \underline{\underline{(\theta_\lambda^\kappa\omega_{\kappa\mu} + \theta_\mu^\kappa\omega_{\kappa\lambda})}}] \quad . \end{aligned} \tag{5.2}$$

With the nonlinear part $N_n^{\alpha\beta}$ of the stress resultant tensor we introduce a nonlinear operator $C(u)$ by defining:

$$C(u) = -\begin{pmatrix} [N_n^{\alpha\beta}(u) - \underline{\frac{1}{2}\omega^{\alpha\beta}(u)N_\lambda^\lambda(u)} - \underline{\underline{\frac{1}{2}(\omega^{\alpha\lambda}(u)N_\lambda^\beta(u)+\omega^{\beta\lambda}(u)N_\lambda^\alpha(u)) + \frac{1}{2}(\theta^{\alpha\lambda}(u)N_\lambda^\beta(u)}} - \\ \underline{\underline{- \theta^{\beta\lambda}(u)N_\lambda^\alpha(u)}}]|_\beta - \underline{b_\beta^\alpha\varphi_\lambda(u)N^{\lambda\beta}(u)} \\ \underline{[\varphi_\alpha(u)N^{\alpha\beta}(u)]|_\beta} + b_{\alpha\beta}[N_n^{\alpha\beta}(u) - \underline{\underline{\omega^{\alpha\lambda}(u)N_\lambda^\beta(u)}}] \end{pmatrix} \tag{5.3}$$

Associated with the operator $C(u)$ we introduce a nonlinear boundary operator $c(u)$ given by:

$$c(u) = \begin{pmatrix} \nu_\alpha\nu_\beta[N_n^{\alpha\beta}(u) - \underline{\underline{\omega^{\alpha\lambda}(u)N_\lambda^\beta(u)}}] \\ t_\alpha\nu_\beta[N_n^{\alpha\beta}(u) - \underline{\frac{1}{2}\omega^{\alpha\beta}(u)N_\lambda^\lambda(u)} - \underline{\underline{\frac{1}{2}(\omega^{\alpha\lambda}(u)N_\lambda^\beta(u) + \omega^{\beta\lambda}(u)N_\lambda^\alpha(u)) +}} \\ \underline{\underline{+\frac{1}{2}(\theta^{\alpha\lambda}(u)N_\lambda^\beta(u) - \theta^{\beta\lambda}(u)N_\lambda^\alpha(u))}}] \\ \underline{\nu_\beta\varphi_\alpha(u)N^{\alpha\beta}(u)} \\ 0 \\ 0 \end{pmatrix} \tag{5.4}$$

With the linear operators of section 4 and the nonlinear operators (5.3) and (5.4) we formulate the nonlinear shell boundary value problem of section 2 in operator description:

$$\begin{aligned} Au + C(u) &= p && \text{in } M \\ \pi u &= u_b^* && \text{on } C_u \\ au + c(u) &= p_b^* && \text{on } C_f \quad , \end{aligned} \tag{5.5}$$

where $(5.5)_1$ are the nonlinear equilibrium equations (2.8) expressed in displacements, $(5.5)_2$ are the geometric boundary conditions (2.13) and $(5.5)_3$ are the nonlinear static boundary conditions (2.14) in displacements.

To derive the governing equations of the buckling and post-buckling analysis in a unified and compact form we introduce a nonlinear row operator $P(u)$ defined by:

$$P(u) = \begin{bmatrix} Au + C(u) & \text{in } M \\ au + c(u) & \text{on } C_u \\ au + c(u) & \text{on } C_f \end{bmatrix} . \tag{5.6}$$

The given middle surface forces p and the given statical boundary variables P_b^* are contracted to a given row vector F

$$F = \begin{bmatrix} p & \text{in } M \\ o & \text{on } C_u \\ P_b^* & \text{on } C_f \end{bmatrix} \tag{5.7}$$

Furthermore we introduce a bilinear form $\{u,p\}$ by

$$\{u,p\} = (u,p) + [u_b,P_b] . \tag{5.8}$$

It will be shown in the following section that with (5.6) - (5.8) the nonlinear shell boundary value problem (5.5) can be given in a variational formulation of the type $\{P(u) - F,\hat{u}\} = o \ , \ \forall \hat{u} \in \overset{o}{H}_u$.

6. VARIATIONAL PRINCIPLE OF TOTAL POTENTIAL ENERGY FOR THE NONLINEAR SHELL BOUNDARY VALUE PROBLEM

As solution of the nonlinear shell boundary value problem (5.5) displacement fields $\bar{u}$ are obtained, which, as function of a one-parametric load (5.7), form a nonlinear equilibrium path, where points of bifurcation or limit load points can occur. At bifurcation points the fundamental equilibrium path is intersected by a bifurcated path which can be stable or unstable. At limit load points the deformation of the shell structure becomes unstable. In this section a generalized Taylor expansion of the total potential energy of the shell is considered, which leads to a variational formulation of the non-linear shell boundary value problem.

Let $\hat{u}$ denote the displacement field from a configuration $\bar{u}$ to an adjacent configuration $\bar{u} + \hat{u}$:

$$u = \bar{u} + \hat{u} \ , \tag{6.1}$$

where it is assumed that $\bar{u}$ satisfies the geometric boundary conditions $(5.5)_2$ and that $\hat{u}$ satisfies the homogeneous geometric boundary conditions $\pi\hat{u}_b = 0$ on C_u. All quantities will be referred to the known undeformed shell middle surface, which allows to derive the shell equations in a Lagrangean description. For the linearized field variables (2.3) and $(2.4)_2$ of the adjacent state we have by superposition:

$$\theta_{\alpha\beta} = \bar{\theta}_{\alpha\beta} + \hat{\theta}_{\alpha\beta} \ ; \ \omega_{\alpha\beta} = \bar{\omega}_{\alpha\beta} + \hat{\omega}_{\alpha\beta} \ ; \ \varphi_i = \bar{\varphi}_i + \hat{\varphi}_i \ ; \ \kappa_{\alpha\beta} = \bar{\kappa}_{\alpha\beta} + \hat{\kappa}_{\alpha\beta} . \tag{6.2}$$

Inserting $(6.2)_1$ and $(6.2)_2$ into $(2.4)_1$ the middle surface strain tensor $\gamma_{\alpha\beta}$ is obtained in the following form:

$$\gamma_{\alpha\beta} = \bar{\gamma}_{\alpha\beta} + \underline{\frac{1}{2}(\bar{\varphi}_\alpha\hat{\varphi}_\beta + \hat{\varphi}_\alpha\bar{\varphi}_\beta)} + \underline{a_{\alpha\beta}\bar{\varphi}_3\hat{\varphi}_3} + \hat{\gamma}_{\alpha\beta} - \underline{\underline{\frac{1}{2}(\bar{\theta}^\lambda_\alpha\hat{\omega}_{\lambda\beta} + \hat{\theta}^\lambda_\alpha\bar{\omega}_{\lambda\beta} + \bar{\theta}^\lambda_\beta\hat{\omega}_{\lambda\alpha} + \hat{\theta}^\lambda_\beta\bar{\omega}_{\lambda\alpha})}} \tag{6.3}$$

where $\bar{\gamma}_{\alpha\beta}$ and $\hat{\gamma}_{\alpha\beta}$ are given by $(2.4)_1$ with corresponding indications.

The total potential energy $J_p(u)$ of the moderate rotation shell theory is described by the functional [2,3]

$$J_p(u) = \iint_M \{f[\gamma_{\alpha\beta}(u) \ , \kappa_{\alpha\beta}(u)] - p \cdot u\} dS - \int_{C_f} (P^* \cdot u + M^*_{\nu\nu}\beta_\nu) ds - \sum_{j=1}^{N_f} F^*_j u_{3j} \tag{6.4}$$

where $f[\gamma_{\alpha\beta}(u), \kappa_{\alpha\beta}(u)]$ is the strain energy density per unit area of the undeformed shell middle surface, given by:

$$f(u) = \frac{1}{2} H^{\alpha\beta\lambda\mu} [\gamma_{\alpha\beta}(u)\gamma_{\lambda\mu}(u) + \frac{h^2}{12} \kappa_{\alpha\beta}(u)\kappa_{\lambda\mu}(u)] \ . \tag{6.5}$$

The first integral of (6.4) is taken over the undeformed shell middle surface M, the second integral along the boundary line C_f, and the third term is the contribution of all corner points of C_f.

The energy increment $\Delta J_p(\bar{u};\hat{u})$ from a fundamental state $\bar{u}$ to an adjacent state $\bar{u} + \hat{u}$ is a measure for the behaviour of the shell structure at $\bar{u}$. It can be derived by using a generalized Taylor expansion of the functional (6.4) leading to the series:

$$\begin{aligned} \Delta J_p(\bar{u};\hat{u}) &= J_p(\bar{u} + \hat{u}) - J_p(\bar{u}) \\ &= J'_p(\bar{u};\hat{u}) + \frac{1}{2} J''_p(\bar{u};\hat{u}\ \hat{u}) + \frac{1}{6} J'''_p(\bar{u};\hat{u}\ \hat{u}\ \hat{u}) + \frac{1}{24} J''''_p(\bar{u};\hat{u}\ \hat{u}\ \hat{u}\ \hat{u}) \ , \end{aligned} \tag{6.6}$$

where J'_p, J''_p, J'''_p, J''''_p denote the first, second, third and fourth Gâteaux differentials of the total potential energy J_p. They can be expressed in the following form by using the differentials of the strain energy density (6.5):

$$\begin{aligned} J'_p(\bar{u};\hat{u}) &= \iint_M [f'(\bar{u};\hat{u}) - p \cdot \hat{u}] ds - \int_{C_f} (P^*_\nu \cdot \hat{u}_\nu + M^*_{\nu\nu}\hat{\beta}_\nu) ds - \sum_{j=1}^{N_f} F^*_j \hat{u}_{3j} \\ J''_p(\bar{u};\hat{u}\ \hat{u}) &= \iint_M f''(\bar{u};\hat{u}\ \hat{u}) ds = J''_p(\bar{u}\ ;\hat{u}^2) \\ J'''_p(\bar{u};\hat{u}\ \hat{u}\ \hat{u}) &= \iint_M f'''(\bar{u};\hat{u}\ \hat{u}\ \hat{u}) ds = J'''_p(\bar{u}\ ;\hat{u}^3) \\ J''''_p(\bar{u};\hat{u}\ \hat{u}\ \hat{u}\ \hat{u}) &= \iint_M f''''(\bar{u};\hat{u}\ \hat{u}\ \hat{u}\ \hat{u}) ds = J''''_p(\bar{u}\ ;\hat{u}^4) \end{aligned} \tag{6.7}$$

All higher order differentials disappear.

To calculate the first differential $f'(\bar{u};\hat{u})$, we have to consider the expression

$$\lim_{t \to o} \frac{f(\bar{u} + t\hat{u}) - f(\bar{u})}{t} = f'(\bar{u};\hat{u}) \tag{6.8}$$

and if this limes is linear in $\hat{u}$, it is called the first Gâteaux differential of the strain energy density $f(u)$.

Let $\hat{u}$ and $\check{u}$ be two different displacement fields superimposed upon the fundamental displacement field $\bar{u}$. The second differential of (6.5) is then defined by the limes:

$$\lim_{t \to o} \frac{f'(\bar{u} + t\check{u};\hat{u}) - f'(\bar{u};\hat{u})}{t} = f''(\bar{u};\check{u}\ \hat{u}) \tag{6.9}$$

Differentials of the third and fourth order have to be calculated analogously.

Introducing the strain tensor $\gamma_{\alpha\beta}(u)$ and the change of curvature tensor $\kappa_{\alpha\beta}(u)$ according to (2.4) into (6.5), the differentials of $f(u)$ can be determined:

$$f'(\bar{u};\hat{u}) = H^{\alpha\beta\lambda\mu}[\bar{\gamma}_{\alpha\beta}\gamma'_{\lambda\mu}(\bar{u};\hat{u}) + \frac{h^2}{12}\bar{\kappa}_{\alpha\beta}\hat{\kappa}_{\lambda\mu}] =$$

$$= \bar{N}^{\alpha\beta}\gamma'_{\alpha\beta}(\bar{u};\hat{u}) + \bar{M}^{\alpha\beta}\hat{\kappa}_{\alpha\beta}$$

$$f''(\bar{u};\check{u}\ \hat{u}) = H^{\alpha\beta\lambda\mu}[\gamma'_{\alpha\beta}(\bar{u};\check{u})\gamma'_{\lambda\mu}(\bar{u}\ ;\ \hat{u}) + \bar{\gamma}_{\alpha\beta}\gamma''_{\lambda\mu}(\check{u},\hat{u}) + \frac{h^2}{12}\check{\kappa}_{\alpha\beta}\hat{\kappa}_{\lambda\mu}] \tag{6.10}$$

$$f'''(\bar{u};\check{u}\ \check{u}\ \hat{u}) = H^{\alpha\beta\lambda\mu}[2\gamma'_{\alpha\beta}(\bar{u};\check{u})\gamma''_{\lambda\mu}(\check{u},\hat{u}) + \gamma'_{\alpha\beta}(\bar{u};\hat{u})\gamma''_{\lambda\mu}(\check{u},\check{u})]$$

$$f''''(\bar{u}\ ;\ \check{u}\ \check{u}\ \check{u}\ \hat{u}) = 3H^{\alpha\beta\lambda\mu}\gamma''_{\alpha\beta}(\check{u},\check{u})\gamma''_{\lambda\mu}(\check{u},\hat{u}) \quad ,$$

where the following differentials of the strain tensor $\gamma_{\alpha\beta}(u)$ are used:

$$\gamma'_{\alpha\beta}(\bar{u};\hat{u}) = \hat{\theta}_{\alpha\beta} + \underline{\frac{1}{2}(\bar{\varphi}_\alpha\hat{\varphi}_\beta + \hat{\varphi}_\alpha\bar{\varphi}_\beta)} + \underline{a_{\alpha\beta}\bar{\varphi}_3\hat{\varphi}_3} - \underline{\underline{\frac{1}{2}(\bar{\theta}^\kappa_\alpha\hat{\omega}_{\kappa\beta} + \hat{\theta}^\kappa_\alpha\bar{\omega}_{\kappa\beta})}} -$$

$$- \underline{\underline{\frac{1}{2}(\bar{\theta}^\kappa_\beta\hat{\omega}_{\kappa\alpha} + \hat{\theta}^\kappa_\beta\bar{\omega}_{\kappa\alpha})}} \tag{6.11}$$

$$\gamma''_{\alpha\beta}(\check{u},\hat{u}) = \underline{\frac{1}{2}(\check{\varphi}_\alpha\hat{\varphi}_\beta + \hat{\varphi}_\alpha\check{\varphi}_\beta)} + \underline{a_{\alpha\beta}\check{\varphi}_3\hat{\varphi}_3} - \underline{\underline{\frac{1}{2}(\check{\theta}^\kappa_\alpha\hat{\omega}_{\kappa\beta} + \hat{\theta}^\kappa_\alpha\check{\omega}_{\kappa\beta}) - \frac{1}{2}(\check{\theta}^\kappa_\beta\hat{\omega}_{\kappa\alpha} + \hat{\theta}^\kappa_\beta\check{\omega}_{\kappa\alpha})}}$$

With (6.7), (6.10) and (6.11) the Taylor expansion (6.6) of the energy increment $\Delta J_p(\bar{u};\hat{u})$ in the neighbourhood of a fundamental state $\bar{u}$ is given.

The principle of total potential energy yields the following statement. For all geometrically admissible displacement fields u, satisfying the geometric boundary conditions $(5.5)_2$ the total potential energy (6.4) attains its stationary value

$$J_p'(u;\hat{u}) = 0 \qquad \forall \hat{u} \in \overset{\circ}{H}_u \; ; \qquad \pi u = u_b^* \tag{6.12}$$

at the solution $u = \bar{u}$. With $\overset{\circ}{H}_u$ we denote a Hilbert-space of sufficiently differentiable displacement fields satisfying homogeneous geometric boundary conditions on C_u.

To derive the Euler-Lagrange equations associated with the variational statement (6.12), we introduce $(6.10)_1$ into $(6.7)_1$ and transform $J_p'(u;\hat{u})$ by partial integration leading to the expression:

$$\begin{aligned}
J_p'(u;\hat{u}) = -\iint_M \{ [(& N^{\alpha\beta} - \tfrac{1}{2}(b^\alpha_\lambda M^{\lambda\beta} + b^\beta_\lambda M^{\lambda\alpha}) - \tfrac{1}{2}(b^\alpha_\lambda M^{\lambda\beta} - b^\beta_\lambda M^{\lambda\alpha}) - \tfrac{1}{2}\,\omega^{\alpha\beta} N^\lambda_\lambda \\
& - \tfrac{1}{2}(\omega^{\alpha\lambda} N^\beta_\lambda + \omega^{\beta\lambda} N^\alpha_\lambda) + \tfrac{1}{2}(\theta^{\alpha\lambda} N^\beta_\lambda - \theta^{\beta\lambda} N^\alpha_\lambda))|_\beta \\
& - b^\alpha_\beta (M^{\lambda\beta}|_\lambda + \varphi_\lambda N^{\lambda\beta}) + p^\alpha] \hat{u}_\alpha \\
& + [(M^{\alpha\beta}|_\alpha + \varphi_\alpha N^{\alpha\beta})|_\beta + b_{\alpha\beta}(N^{\alpha\beta} - b^\alpha_\lambda M^{\lambda\beta} - \omega^{\alpha\lambda} N^\beta_\lambda) + p_3] \hat{u}_3 \} dS \\
+ \int_C \{ [\nu_\alpha \nu_\beta (& N^{\alpha\beta} - b^\alpha_\lambda M^{\lambda\beta} - \omega^{\alpha\lambda} N^\beta_\lambda) - b_{\nu t} M_{t\nu} - P^*_\nu] \hat{u}_\nu \\
& + [t_\alpha \nu_\beta (N^{\alpha\beta} - \tfrac{1}{2}(b^\alpha_\lambda M^{\lambda\beta} + b^\beta_\lambda M^{\lambda\alpha}) - \tfrac{1}{2}(b^\alpha_\lambda M^{\lambda\beta} - b^\beta_\lambda M^{\lambda\alpha}) - \tfrac{1}{2}\,\omega^{\alpha\beta} N^\lambda_\lambda \\
& - \tfrac{1}{2}(\omega^{\alpha\lambda} N^\beta_\lambda + \omega^{\beta\lambda} N^\alpha_\lambda) + \tfrac{1}{2}(\theta^{\alpha\lambda} N^\beta_\lambda - \theta^{\beta\lambda} N^\alpha_\lambda)) - b_{tt} M_{t\nu} - P^*_t] \hat{u}_t \\
& + [\nu_\beta (M^{\alpha\beta}|_\alpha + \varphi_\alpha N^{\alpha\beta}) + \frac{d}{ds} M_{t\nu} - P^*_3] \hat{u}_3 + (M_{\nu\nu} - M^*_{\nu\nu}) \hat{\beta}_\nu \} ds \\
+ \sum_{j=1}^{N_f} & [M_{t\nu}(s_{fj} + 0) - M_{t\nu}(s_{fj} - 0) - F^*_j] \hat{u}_3 \; ,
\end{aligned} \tag{6.13}$$

where sufficient differentiability of u is assumed. Introducing (6.13) into the stationarity condition (6.12) yields the nonlinear equilibrium equations (2.8) and the nonlinear static boundary conditions (2.14) as Euler-Lagrange equations.

To derive the Euler-Lagrange equations in the form (5.5) the membrane stress tensor $N^{\alpha\beta}$ is splitted into the linear part $N_{\ell}^{\alpha\beta}$ and the nonlinear part $N_{n}^{\alpha\beta}$ with (5.1) and (5.2). Using the bilinear forms and operators of section 4 and 5, the first differential of the total potential energy (6.13) can be expressed by:

$$J_p'(u;\hat{u}) = (Au + C(u) - p, \hat{u}) + [au + c(u), \hat{u}_b]_1 + [au + c(u) - P_b^*, \hat{u}_b]_2$$

$$= \{P(u) - F, \hat{u}\} \quad . \qquad (6.14)$$

With (6.14) the stationarity condition (6.12) is obtained in the compact from:

$$\{P(u) - F, \hat{u}\} = 0 \qquad \forall \hat{u} \in \overset{\circ}{H}_u \quad ; \qquad \pi u = u_b^* \qquad (6.15)$$

with the associated Euler-Lagrange equations:

$$\begin{aligned} Au + C(u) &= p \quad \text{in } M \\ au + c(u) &= P_b^* \quad \text{on } C_f \end{aligned} \qquad (6.16)$$

defining the nonlinear elastic shell boundary value problem.

In the following sections it will be shown that the equations of critical equilibrium, defining snap-through and bifurcation buckling, and also the post-buckling equations can be given in a compact form by using the differentials of the operators P and A, C, a, c, respectively. All differentials of these operators are given in the appendix for the nonlinear elastic shell theory with moderate rotations.

To solve nonlinear shell boundary value problems by using the stationary principle of total potential energy we can proceed as follows. Let $\tilde{u}$ denote geometrically admissible displacement fields of the shell middle surface, satisfying the geometric boundary conditions (2.13). If we choose any geometrically admissible displacement $\tilde{u}_o$, we obtain by superposition:

$$u = \tilde{u}_o + \hat{u} \quad ; \quad \pi\tilde{u}_o = u^*_b \quad ; \quad \hat{u} \in \mathring{H}_u \ , \tag{6.17}$$

where the elements $\hat{u}$ of the infinite dimensional space $\mathring{H}_u$ satisfy homogeneous geometric boundary conditions. For arbitrary $\hat{u} \in \mathring{H}_u$ the total potential energy $J_p(u) = J_p(\tilde{u}_o + \hat{u})$ attains its stationary value at the solution point $u_o = \tilde{u}_o + \hat{u}_o$ of the corresponding nonlinear shell boundary value problem.

For a numerical application the infinite dimensional problem can be reduced to a finite dimensional problem by discretization using approximation procedures as a Rayleigh-Ritz or equivalent finite element method. They enable to construct a finite dimensional subset of $\mathring{H}_u$ by choosing n linearly independent coordinate functions with unknown coefficients q_i, $i = 1,\dots,n$, where the coordinate functions have to be complete in $\mathring{H}_u$ to assure convergence of the approximation procedure. The stationarity condition for the functional $J_p(q_i)$, $i = 1,\dots,n$ yields a system of n inhomogeneous nonlinear equations in the unknown coefficients q_i. The solution of these equations leads to an approximated displacement field of the shell middle surface, converging for $n \to \infty$ towards the exact solution of the nonlinear shell boundary value problem.

7. APPROACH BY A SEQUENCE OF LINEAR SHELL BOUNDARY VALUE PROBLEMS

The total potential energy $J_p(u,F)$ is a functional of the displacement field u and of the external load F according to (5.7), where F may depend on various load parameters λ_j, $j = 1,\dots,m$. In the special case of a nonlinear elastic plate, loaded by two independent compressive boundary forces, P_1 and P_2, F depends linearly on two load parameters λ_1 and λ_2. This problem is investigated in [23], where also a prescribed relationship between λ_1 and λ_2 and another load parameter λ is assumed.

For many stability problems of engineering interest we can restrict our considerations to the study of shell deformations along a fundamental equilibrium path, described by a one-parameter dependency

$$\lambda_j = \lambda_j(\lambda) \quad , \quad j = 1,\dots,m \, , \tag{7.1}$$

where the external load F depends linearly on the load parameter λ

$$F = \lambda f \qquad (7.2)_1$$

with f denoting a unit load system:

$$f = \begin{Bmatrix} p_* & \text{in } M \\ 0 & \text{on } C_u \\ p_{b*} & \text{on } C_f \end{Bmatrix} \quad . \qquad (7.2)_2$$

Using (6.12) and (6.15) the stationarity condition for the fundamental path can be given in the two alternative forms:

$$\begin{aligned} &J_p'(\bar{u}(\lambda)\, ; \hat{u}) = 0 && \forall \hat{u} \in \overset{\circ}{H}_u \\ &\{P(\bar{u}(\lambda)) - \lambda f \, , \hat{u}\} = 0 && \forall \hat{u} \in \overset{\circ}{H}_u \end{aligned} \tag{7.3}$$

We consider now the problem of solving (7.3) in the neighbourhood of a solution point 0 with known displacement field u_o by a linear analysis.

Introducing a parameter t with t = o at O, it is assumed that the solution of (7.3) can be expressed by the parametric expansion

$$\begin{aligned}\bar{u}(t) &= u_o + \check{u}(t) \\ &= u_o + t\dot{u}_o + \frac{t^2}{2}\ddot{u}_o + \frac{t^3}{6}\dddot{u}_o + O(t^4)\end{aligned} \tag{7.4}$$

where a dot indicates differentiation with respect to the parameter t.

An expansion of the same type is used to represent the load parameter $\lambda(t)$:

$$\lambda(t) = \lambda_o + t\dot{\lambda}_o + \frac{t^2}{2}\ddot{\lambda}_o + \frac{t^2}{6}\dddot{\lambda}_o + O(t^4)\ . \tag{7.5}$$

With (7.4) the governing shell operator $P(u(\lambda))$ according to (5.6) can be expanded in a Taylor series:

$$\begin{aligned}P(\bar{u}(t)) &= P(u_o + \check{u}(t)) \\ &= P(u_o) + tP'(u_o)\dot{u}_o + \frac{t^2}{2}[P'(u_o)\ddot{u}_o + P''(u_o)\dot{u}_o^2] \\ &\quad + \frac{t^3}{6}[P'(u_o)\dddot{u}_o + 3P''(u_o)\dot{u}_o\ddot{u}_o + P'''(u_o)\dot{u}_o^3] + O(t^4)\ .\end{aligned} \tag{7.6}$$

Introducing (7.5) and (7.6) into the stationarity condition $(7.3)_2$ the following sequential variational statements are obtained:

$$\begin{aligned}&\{P(u_o) - \lambda_o \mathfrak{f}, \hat{u}\} = o && \forall \hat{u} \\ &\{P'(u_o)\dot{u}_o - \dot{\lambda}_o \mathfrak{f}, \hat{u}\} = o && \forall \hat{u} \\ &\{P'(u_o)\ddot{u}_o + P''(u_o)\dot{u}_o^2 - \ddot{\lambda}_o \mathfrak{f}, \hat{u}\} = o && \forall \hat{u} \\ &\{P'(u_o)\dddot{u}_o + 3P''(u_o)\dot{u}_o\ddot{u}_o + P'''(u_o)\dot{u}_o^3 - \dddot{\lambda}_o \mathfrak{f}, \hat{u}\} = o && \forall \hat{u} \\ &\vdots\end{aligned} \tag{7.7}$$

The sequential stationarity conditions (7.7) define a sequence of shell boundary value problems, which allow to determine the unknown coefficients of expansion (7.4). The Euler-Lagrange equations corresponding to $(7.7)_1$ are the nonlinea equilibrium equations and the nonlinear static boundary conditions

$$\begin{aligned} Au_o + C(u_o) &= \lambda_o p_* \quad \text{in } M \\ au_o + c(u_o) &= \lambda_o p_{b*} \quad \text{on } C_f \end{aligned} \tag{7.8}$$

with the solution u_o.

We assume that the solution point 0 is a regular point with a unique solution u_o of (7.8). Then $(7.7)_2$ yields the linear equations:

$$\begin{aligned} A\dot{u}_o + C'(u_o)\dot{u}_o &= \dot{\lambda}_o p_* \quad \text{in } M \\ a\dot{u}_o + c'(u_o)\dot{u}_o &= \dot{\lambda}_o p_{b*} \quad \text{on } C_f \end{aligned} \tag{7.9}$$

depending nonlinearly on u_o. The solution of this linear boundary value problem is the displacement field $\dot{u}_o$.

The variational statement $(7.7)_3$ leads to the equations

$$\begin{aligned} A\ddot{u}_o + C'(u_o)\ddot{u}_o &= \ddot{\lambda}_o p_* - C''(u_o)\dot{u}_o^2 \quad \text{in } M \\ a\ddot{u}_o + c'(u_o)\ddot{u}_o &= \ddot{\lambda}_o p_{b*} - c''(u_o)\dot{u}_o^2 \quad \text{on } C_f \end{aligned} \tag{7.10}$$

with the solution $\ddot{u}_o$. Correspondingly the variational statements $(7.7)_4$, $(7.7)_5$,... define linear shell boundary value problems with the solutions $\dddot{u}_o$, $\ddddot{u}_o$, ..., the coefficients of expansion (7.4). If this approximation procedure is stopped after n terms the equilibrium error can be calculated, what will not be considered here because of lack of space.

8. CRITICAL POINTS OF THE FUNDAMENTAL PATH

The determination of the critical points of the fundamental path can be performed by considering the behaviour of the total potential energy $J_p(u(t), \lambda(t))$ in the vicinity of a solution point u_M, λ_M along every possible one-parameter deformation $\check{u}(t)$

$$u(t) = u_M + \check{u}(t)$$

$$\check{u}(t) = tu^{\cdot} + \frac{t^2}{2}u^{\cdot\cdot} + \frac{t^3}{6}u^{\cdot\cdot\cdot} + o(t^4) \tag{8.1}$$

$$\lambda = \lambda(t) = \lambda_M$$

superimposed upon the deformation u_M, where the load parameter λ retains the constant value λ_M.

The energy increment (6.6) at the solution point u_M can be obtained as Taylor expansion:

$$\Delta J_p(u_M ; \check{u}(t)) = \frac{t^2}{2} J_p''(u_M ; u^{\cdot 2}) + \frac{t^3}{6}[3J_p''(u_M ; u^{\cdot}u^{\cdot\cdot}) + J_p'''(u_M ; u^{\cdot 3})] +$$

$$+ \frac{t^4}{24}[J_p''(u_M ; u^{\cdot}u^{\cdot\cdot\cdot}) + 3J_p''(u_M ; u^{\cdot\cdot 2}) + 6J_p'''(u_M ; u^{\cdot 2}u^{\cdot\cdot}) +$$

$$+ J_p''''(u_M ; u^{\cdot 4})] + o(t^5) \quad . \tag{8.2}$$

It will be shown that at critical points the first term on the right side of (8.2) will vanish.

We have to investigate whether the one-parameter deformation (8.1) can be also an equilibrium deformation, such that the sequential stationarity conditions (7.7) are satisfied for constant λ - values. In this case all derivatives of λ vanish and (7.7) yields:

$$\{P(u_M) - \lambda_M \delta , \hat{u}\} = 0 \qquad \forall \hat{u}$$

$$\{P'(u_M)u^{\cdot} , \hat{u}\} = 0 \qquad \forall \hat{u}$$

$$\{P'(u_M)u^{\cdot\cdot} + P''(u_M)u^{\cdot 2} , \hat{u}\} = 0 \qquad \forall \hat{u} \tag{8.3}$$

$$\{P'(u_M)u^{\cdot\cdot\cdot} + 3P''(u_M)u^{\cdot}u^{\cdot\cdot} + P'''(u_M)u^{\cdot 3} , \hat{u}\} = 0 \qquad \forall \hat{u}$$

$$\vdots$$

The variational statement $(8.3)_1$ is the stationarity condition for the solution point u_M, λ_M, which is also the trivial solution of $(8.3)_2$, $(8.3)_3$,... for $u^{\bullet} = u^{\bullet\bullet} = \ldots = 0$.

If we compare $(8.3)_2$ with $(7.7)_2$ it is obvious that $(8.3)_2$ has no solution in the non-critical case and therefore also $(8.3)_3$, ... cannot be satisfied. Only in the critical case, denoted by

$$\lambda_M = \lambda_c \quad ; \; u_M = u_c \qquad (8.4)_1$$

and satisfying the stationarity condition

$$\{P(u_c) - \lambda_c \, \delta, \hat{u}\} = 0 \qquad \forall \hat{u} \qquad (8.4)_2$$

there may be one or more solutions of the variational statement:

$$\{P'(u_c)u_c^{\bullet}, \hat{u}\} = 0 \qquad \forall \hat{u} \qquad (8.5)$$

with the restriction that $u_c^{\bullet}$ has to satisfy some normalizing condition.

The solution equations of (8.5) yield the conditions for critical equilibrium, the equations of snap-through or bifurcation buckling of shells:

$$\begin{aligned} A u_c^{\bullet} + C'(u_c) u_c^{\bullet} &= 0 \qquad \text{in } M \\ a u_c^{\bullet} + c'(u_c) u_c^{\bullet} &= 0 \qquad \text{on } C_f \end{aligned} \qquad (8.6)$$

with the linear operators A and a, defined by (4.8) and (4.13), and the non-linear operators C' and c', given in the appendix by (A.7) and (A.10). The linear equation $(8.6)_1$ represents three homogeneous field equations of the shell middle surface and $(8.6)_2$ four associated static boundary conditions. They define a homogeneous eigenvalue problem depending nonlinearly on the critical pre-buckling deformation $u_c(\lambda_c)$, where $u_c(\lambda_c)$ has to satisfy the stationarity condition $(8.4)_2$. The solution of the eigenvalue problem (8.5) and (8.6), respectively, yields the buckling mode $u_c^{\bullet}$ with the associated buckling load λ_c.

To normalize the buckling mode $u^{\bullet}_{c}$ we choose a normalization according to the strain energy of linear shell deformations by setting:

$$\{P'(0)u^{\bullet}_{c}, u^{\bullet}_{c}\} = 1 \quad . \tag{8.7}$$

Here $P'(0)$ denotes the derivative of the operator $P(u)$ at $u = 0$. Using definition (5.6) of the operator $P(u)$ and definition (5.8) of the bilinear form $\{.,.\}$, expression (8.7) represents the double elastic shell energy for the small deformation $u^{\bullet}_{c}$:

$$\{P'(0)u^{\bullet}_{c}, u^{\bullet}_{c}\} = (Au^{\bullet}_{c}, u^{\bullet}_{c}) + [au^{\bullet}_{c}, u^{\bullet}_{bc}] =$$

$$= \iint_{M} H^{\alpha\beta\lambda\mu}[\theta^{\alpha\beta}(u^{\bullet}_{c})\theta_{\lambda\mu}(u^{\bullet}_{c}) + \frac{h^2}{12}\kappa_{\alpha\beta}(u^{\bullet}_{c})\kappa_{\lambda\mu}(u^{\bullet}_{c})]ds \; . \tag{8.8}$$

Using $u^{\bullet}_{c}$ to define shell buckling modes, higher order contributions of expansion $(8.1)_2$ are not considered and the question, whether or not $(8.3)_3, \ldots$ are satisfied, is not discussed.

Using formula $(A.13)_2$ expression (8.5) can be transformed by partial integration. Choosing then $\hat{u} = u^{\bullet}_{c} \in \overset{\circ}{H}_{u}$ we obtain:

$$\{P'(u_c)u^{\bullet}_{c}, u^{\bullet}_{c}\} = J''_{p}(u_c ; u^{\bullet 2}_{c}) = 0 \; , \tag{8.9}$$

leading to the result that the first term of the energy expansion (8.2) vanishes. This corresponds to the energy criterion of stability that for critical deformations the second differential of the total potential energy must disappear.

If we compare the stability equations (8.6) with the governing equations $(5.5)_1$ and $(5.5)_3$ of the nonlinear shell boundary value problem, it is obvious that (8.6) is the differential of $(5.5)_1$ and $(5.5)_3$ for $\lambda(t) = \lambda_c$. This gives a very general form of the so-called adjacent equilibrium criterion well-known in the engineering literature [41].

9. BIFURCATION AND POST-BUCKLING ANALYSIS FOR SINGLE BUCKLING MODES

Let us assume that, at $\lambda = \lambda_c$, a new equilibrium path $u^b(\lambda)$ bifurcates from the fundamental path $\bar{u}(\lambda)$:

$$u^b(\lambda) = \bar{u}(\lambda) + v(\lambda) \quad . \tag{9.1}$$

With $v(\lambda)$ we denote the differential displacement field from the fundamental to the bifurcated path. Under suitable regularity conditions the fundamental equilibrium path $\bar{u}(\lambda)$ can be expressed by the Taylor expansion near the bifurcation point u_c, λ_c:

$$\bar{u}(\lambda) = u_c + (\lambda - \lambda_c)u_c^{\Diamond} + \frac{1}{2}(\lambda - \lambda_c)^2 u_c^{\Diamond\Diamond} + \frac{1}{6}(\lambda - \lambda_c)^3 u_c^{\Diamond\Diamond\Diamond} + O((\lambda - \lambda_c)^4) \quad , \tag{9.2}$$

where the symbol $(\ )^{\Diamond}$ indicates differentiation with resprect to the load parameter λ, $(\ )^{\Diamond} = \frac{d(\ )}{d\lambda}$. The index c refers to the bifurcation point.

For the differential displacement $v(\lambda(t))$ we use the parametric representation with (t = 0) at the bifurcation point:

$$v(t) = t\dot{v}_c + \frac{t^2}{2}\ddot{v}_c + \frac{t^3}{6}\dddot{v}_c + O(t^4) \quad . \tag{9.3}$$

The corresponding expansion of the load parameter $\lambda(t)$ near t = 0 yields:

$$\lambda(t) = \lambda_c + t\dot{\lambda}_c + \frac{t^2}{2}\ddot{\lambda}_c + \frac{t^3}{6}\dddot{\lambda}_c + O(t^4) \quad . \tag{9.4}$$

The coefficients of the parameter expansion (9.3) may be normalized according to the bilinear form $\{P'(0).,.\}$:

$$\begin{aligned} &\{P'(0)\dot{v}_c , \dot{v}_c\} = 1 \\ &\{P'(0)\dot{v}_c , \ddot{v}_c\} = \{P'(0)\dot{v}_c , \dddot{v}_c\} = \dots = 0 \quad . \end{aligned} \tag{9.5}$$

Assuming that the bifurcated path $u^b(\lambda)$ is an equilibrium path the stationarity condition $(7.3)_2$ must be satisfied:

$$\{P(u^b)) - \lambda f, \hat{u}\} = 0 \qquad \forall \hat{u} \quad . \tag{9.6}$$

With (9.1) we use the Taylor expansion of the operator $P(u^b(\lambda))$:

$$P(u^b(\lambda)) = P(\bar{u}) + P'(\bar{u})v + \frac{1}{2} P''(\bar{u})v^2 + \frac{1}{6} P'''(\bar{u})v^3 \tag{9.7}$$

yielding the two stationarity conditions:

$$\begin{aligned} &\{P(\bar{u}) - \lambda f, \hat{u}\} = 0 && \forall \hat{u} \\ &\{P'(\bar{u})v + \frac{1}{2} P''(\bar{u})v^2 + \frac{1}{6} P'''(\bar{u})v^3, \hat{u}\} = 0 && \forall \hat{u} \end{aligned} \quad . \tag{9.8}$$

Relation $(9.8)_1$ is the stationarity condition for the fundamental path and $(9.8)_2$ the stationarity condition to define the bifurcated path. For further considerations it is presupposed that the solution of $(9.8)_1$, the fundamental path (9.2), is known.

The operators of $(9.8)_2$, $P'(\bar{u})$, $P''(\bar{u})$ and $P'''(\bar{u})$ will be expanded at the bifurcation point. With (9.2) and (9.4) we obtain the series:

$$\begin{aligned} P'(\bar{u}(\lambda(t))) &= P'(u_c) + t\lambda_c^{\bullet} P''(u_c)u_c^{\diamond} + \frac{t^2}{2}[\lambda_c^{\bullet 2} P''(u_c)u_c^{\diamond\diamond} + \lambda_c^{\bullet\bullet} P''(u_c)u_c^{\diamond} + \\ &\qquad + \lambda_c^{\bullet 2} P'''(u_c)u_c^{\diamond 2}] + O(t^3) \\ P''(\bar{u}(\lambda(t))) &= P''(u_c) + t\lambda_c^{\bullet} P'''(u_c)u_c^{\diamond} + O(t^2) \\ P'''(\bar{u}(\lambda(t))) &= P'''(u_c) = P''' \quad . \end{aligned} \tag{9.9}$$

We introduce the operators (9.9) into the stationarity condition $(9.8)_2$. Replacing v by (9.3) and collecting the various terms with respect to the powers of t, the following result can be derived:

$$\begin{aligned} &\{P'(u_c)v_c^{\bullet}, \hat{u}\} = 0 && \forall \hat{u} \\ &\{P'(u_c)v_c^{\bullet\bullet} + P''(u_c)v_c^{\bullet 2} + 2\lambda_c^{\bullet} P''(u_c)u_c^{\diamond} v_c^{\bullet}, \hat{u}\} = 0 && \forall \hat{u} \\ &\{P'(u_c)v_c^{\bullet\bullet\bullet} + 3P''(u_c)v_c^{\bullet}v_c^{\bullet\bullet} + P'''(u_c)v_c^{\bullet 3} + 3\lambda_c^{\bullet}[P''(u_c)u_c^{\diamond}v_c^{\bullet\bullet} + P'''(u_c)u_c^{\diamond}v_c^{\bullet 2}] + \\ &\quad + 3\lambda_c^{\bullet 2}[P''(u_c)u_c^{\diamond\diamond} v_c^{\bullet} + P'''(u_c)u_c^{\diamond 2}v_c^{\bullet}] + 3\lambda_c^{\bullet\bullet} P''(u_c)u_c^{\diamond}v_c^{\bullet}, \hat{u}\} = 0 && \forall \hat{u} \\ &\vdots \end{aligned} \quad , \tag{9.10}$$

representing a sequence of stationarity conditions to determine the bifurcated path (9.1).

The first variational statement $(9.10)_1$ describes together with the normalization condition $(9.5)_1$ the homogeneous eigenvalue problem of bifurcation buckling of thin shells. With definition (5.8) of the bilinear form $\{.,.\}$ and with the operator $P'(u_c)$ according to (A.4) the solution equations of $(9.10)_1$

$$\begin{aligned} A\dot{v}_c + C'(u_c)\dot{v}_c &= 0 \qquad \text{in } M \\ a\dot{v}_c + c'(u_c)\dot{v}_c &= 0 \qquad \text{on } C_f \end{aligned} \tag{9.11}$$

are the stability equations, depending nonlinearly on the prebuckling deformation $u_c(\lambda_c)$. Equation $(9.11)_1$ represents the field equations and $(9.11)_2$ the associated static boundary conditions. The solution of the eigenvalue problem (9.11) yields the buckling mode $\dot{v}_c$ together with the bifurcation load parameter λ_c. In this section it is assumed that λ_c is a singular eigenvalue.

Because of (9.1) the increment of the bifurcated deformation at the bifurcation point is:

$$u^{b\bullet} = \dot{u}_c + \dot{v}_c \quad . \tag{9.12}$$

To determine the coefficient $\dot{\lambda}_c$ of series (9.4), we choose in $(9.10)_2$ $\hat{u} = \dot{v}_c \in \overset{o}{H}_u$. Using the symmetry of the operator $P'(u_c)$, given by $(A.13)_2$, we obtain with $(9.10)_1$:

$$\{P'(u_c)\ddot{v}_c \, , \dot{v}_c\} = \{\ddot{v}_c \, , P'(u_c)\dot{v}_c\} = 0 \tag{9.13}$$

Then $(9.10)_2$ yields for $\hat{u} = \dot{v}_c$:

$$\dot{\lambda}_c = -\frac{1}{2}\,\frac{\{P''(u_c)\dot{v}_c^{2}, \dot{v}_c\}}{\{P''(u_c)u_c^{\Diamond}\dot{v}_c \, , \dot{v}_c\}} \quad , \tag{9.14}$$

where it is assumed that $\{P''(u_c)u_c^{\Diamond}\, \dot{v}_c \, , \dot{v}_c\} \neq 0$.

With known $v_c^{\bullet}$ and $\lambda_c^{\bullet}$ $(9.10)_2$ is a variational statement to determine the coefficient $v_c^{\bullet\bullet}$. The associated solution equations

$$
\begin{aligned}
Av_c^{\bullet\bullet} + C'(u_c)v_c^{\bullet\bullet} &= - C''(u_c)v_c^{\bullet 2} - 2\lambda_c^{\bullet} C''(u_c)u_c^{\Diamond}v_c^{\bullet} \\
av_c^{\bullet\bullet} + c'(u_c)v_c^{\bullet\bullet} &= - c''(u_c)v_c^{\bullet 2} - 2\lambda_c^{\bullet} c''(u_c)u_c^{\Diamond}v_c^{\bullet}
\end{aligned}
\qquad (9.15)
$$

form a set of linear nonhomogeneous partial differential equations, describing a linear shell boundary value problem yielding the displacement $v_c^{\bullet\bullet}$. The first line in (9.15) represents the field equations, the second line the associated static boundary conditions.

The solution of (9.15) is not unique, because any multiple of v_c can be added to a particular solution of (9.15), which we denote by $v_{cp}^{\bullet\bullet}$. To satisfy the normalizing condition $(9.5)_2$ the unique solution $v_c^{\bullet\bullet}$ is given by:

$$
v_c^{\bullet\bullet} = v_{cp}^{\bullet\bullet} - \{P'(0)v_{cp}^{\bullet\bullet}, v_c^{\bullet}\}v_c^{\bullet} \quad , \qquad (9.16)
$$

which can be shown by introducing (9.16) into $(9.5)_2$.

Continuing the procedure we choose in $(9.10)_3$ $\hat{u} = v_c^{\bullet} \in \mathring{H}_u$ such that $\{P'(u_c)v_c^{\bullet\bullet\bullet}, v_c^{\bullet}\}$ disappears because of the symmetry of $P'(u_c)$ and because of $(9.10)_1$. Then the load coefficient $\lambda_c^{\bullet\bullet}$ can be determined from $(9.10)_3$ yielding:

$$
\lambda_c^{\bullet\bullet} = - \frac{\{P''(u_c)v_c^{\bullet}v_c^{\bullet\bullet} + \frac{1}{3}P'''(u_c)v_c^{\bullet 3} + \lambda_c^{\bullet}[P''(u_c)u_c^{\Diamond}v_c^{\bullet\bullet} + P'''(u_c)u_c^{\Diamond}v_c^{\bullet 2}] + \lambda_c^{\bullet 2}[P''(u_c)u_c^{\Diamond\Diamond}v_c^{\bullet} + P'''(u_c)u_c^{\Diamond 2}v_c^{\bullet}], v_c^{\bullet}\}}{\{P''(u_c)u_c^{\Diamond}v_c^{\bullet}, v_c^{\bullet}\}} . \qquad (9.17)
$$

For many shell buckling problems the bifurcation is symmetric. In this case the coefficient $\lambda_c^{\bullet}$ of the load parameter expansion (9.4) vanishes leading to a considerably simplified expression for $\lambda_c^{\bullet\bullet}$:

$$\lambda_c^{\bullet\bullet} = - \frac{\{P''(u_c)v_c^{\bullet}v_c^{\bullet\bullet} + \frac{1}{3}P'''(u_c)v_c^{\bullet 3}, v_c^{\bullet}\}}{\{P''(u_c)u_c^{\Diamond}v_c^{\bullet}, v_c^{\bullet}\}} \quad (9.18)$$

With known $\lambda_c^{\bullet}$, $\lambda_c^{\bullet\bullet}$, $v_c^{\bullet}$, $v_c^{\bullet\bullet}$ expression $(9.10)_3$ is a variational statement to determine the displacement field $v_c^{\bullet\bullet\bullet}$. The associated solution equations define a linear shell boundary value problem, which allows to determine $v_c^{\bullet\bullet\bullet}$. Analogously the further variational statements of (9.10) yield the higher order terms of the series (9.3) and (9.4) describing the bifurcated equilibrium path.

10. POST-BUCKLING ANALYSIS FOR MULTIPLE BUCKLING MODES

The homogeneous eigenvalue problem of shell buckling, described by the variational formulation $(9.10)_1$ or by the stability equations (9.11), may have N linearly independent solutions, the eigenmodes $v_c^{\bullet(i)}$, $i = 1,\dots,N$, associated with a multiple buckling load λ_c:

$$\lambda_c^{(1)} = \lambda_c^{(2)} = \dots = \lambda_c^{(N)} = \lambda_c \quad ; \quad v_c^{\bullet(1)} \; ; \; v_c^{\bullet(2)} ; \; \dots \; , \; v_c^{\bullet(N)} \; . \quad (10.1)$$

We define a multiple buckling mode by

$$v_c^{\bullet} = \sum_{i=1}^{N} a_i v_c^{\bullet(i)} \quad ; \quad \sum_{i=1}^{N} a_i^2 = 1 \quad . \quad (10.2)$$

The eigenmodes $v_c^{\bullet(i)}$ will be normalized according to:

$$\{P'(0)v_c^{\bullet(i)}, v_c^{\bullet(j)}\} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad (10.3)$$

A particular bifurcated path can be defined by the parameter expansion:

$$\begin{aligned} u^b(t) &= \bar{u}(\lambda(t)) + v(t) \\ v(t) &= t a_i v_c^{\bullet(i)} + \frac{t^2}{2} v_c^{\bullet\bullet} + \frac{t^3}{6} v_c^{\bullet\bullet\bullet} + O(t^4) \\ \lambda(t) &= \lambda_c + t\lambda_c^{\bullet} + \frac{t^2}{2}\lambda_c^{\bullet\bullet} + \frac{t^3}{6}\lambda_c^{\bullet\bullet\bullet} + O(t^4) \end{aligned} \quad (10.4)$$

where the summation convention is used for i = 1,...,N. Analog to $(9.5)_2$ the coefficients of $(10.4)_2$ have to satisfy the orthogonality conditions:

$$\{P'(0)a_i v_c^{\bullet(i)}, v_c^{\bullet\bullet}\} = \{P'(0)a_i v_c^{\bullet(i)}, v_c^{\bullet\bullet\bullet}\} = \ldots = 0 \quad . \tag{10.5}$$

If the bifurcated path (10.4) is an equilibrium path, the stationarity condition (9.6) must be satisfied. Following the procedure of the last section and expresssing $P(u^b(\lambda))$ as Taylor expansion along the bifurcated path (10.4) the following sequence of stationarity conditions is obtained:

$$a_i\{P'(u_c)v_c^{\bullet(i)}, \hat{u}\} = 0 \qquad \forall \hat{u}$$

$$\{P'(u_c)v_c^{\bullet\bullet} + a_i a_j P''(u_c)v_c^{\bullet(i)}v_c^{\bullet(j)} + 2\lambda_c^{\bullet} a_i P''(u_c)u_c^{\Diamond}v_c^{\bullet(i)}, \hat{u}\} = 0 \qquad \forall \hat{u}$$

$$\begin{aligned}\{P'(u_c)v_c^{\bullet\bullet\bullet} &+ 3a_i P''(u_c)v_c^{\bullet(i)}v_c^{\bullet\bullet} + a_i a_j a_k P'''(u_c)v_c^{\bullet(i)}v_c^{\bullet(j)}v_c^{\bullet(k)} + \\ &+ 3\lambda_c^{\bullet}[P''(u_c)u_c^{\Diamond}v_c^{\bullet\bullet} + a_i a_j P'''(u_c)u_c^{\Diamond}v_c^{\bullet(i)}v_c^{\bullet(j)}] + \\ &+ 3\lambda_c^{\bullet 2}[a_i P''(u_c)u_c^{\Diamond\Diamond}v_c^{\bullet(i)} + a_i P'''(u_c)u_c^{\Diamond 2}v_c^{\bullet(i)}] + \\ &+ 3\lambda_c^{\bullet\bullet} a_i P''(u_c)u_c^{\Diamond}v_c^{\bullet(i)}, \hat{u}\} = 0 \qquad \forall \hat{u}\end{aligned}$$

$$\vdots \tag{10.6}$$

The statement $(10.6)_1$ is the variational formulation of the eigenvalue problem of shell buckling for multiple buckling modes:

$$\{P'(u_c)v_c^{\bullet(i)}, \hat{u}\} = 0 \qquad \forall \hat{u} \qquad i = 1,\ldots,N \tag{10.7}$$

with the eigenmodes $v_c^{\bullet(i)}$ associated to the multiple buckling load λ_c. Statement (10.7) yields the stability equations of shell buckling:

$$\left.\begin{aligned} Av_c^{\bullet(i)} + C'(u_c)v_c^{\bullet(i)} &= 0 \quad \text{in } M \\ av_c^{\bullet(i)} + c'(u_c)v_c^{\bullet(i)} &= 0 \quad \text{on } C_f \end{aligned}\right\} \quad i = 1,\ldots,N \, . \tag{10.8}$$

To obtain a bifurcated path of the form (10.4) the coefficients $a_i, i=1,\ldots,N$ of expanssion (10.4) must be determined such that the stationarity conditions (10.6) are satisfied. In $(10.6)_2$ we set successively $\hat{u} = v_c^{\bullet(1)}, v_c^{\bullet(2)}, \ldots, v_c^{\bullet(N)}$. Then the first term of $(10.6)_2$ vanishes because of (10.7) and the following set of N nonlinear equations can be derived:

$$a_i a_j \{P''(u_c) v_c^{\bullet(i)} v_c^{\bullet(j)}, v_c^{\bullet k}\} + 2\lambda_c^{\bullet} a_i \{P''(u_c) u_c^{\Diamond} v_c^{\bullet(i)}, v_c^{\bullet(k)}\} = 0, \quad k=1,\ldots,N, \tag{10.9}$$

which has to be solved together with equation $(10.2)_2$ yielding the coefficients $a_1,\ldots,a_N$ and $\lambda_c^{\bullet}$. The load coefficient $\lambda_c^{\bullet}$ is obtained as:

$$\lambda_c^{\bullet} = -\frac{a_i a_j a_k \{P''(u_c) v_c^{\bullet(i)} v_c^{\bullet(j)}, v_c^{\bullet(k)}\}}{2a_i a_j \{P''(u_c) u_c^{\Diamond} v_c^{\bullet(i)}, v_c^{\bullet(j)}\}} . \tag{10.10}$$

There may be many different sets of solutions corresponding to different bifurcated paths. For each set $(10.6)_2$ is a variational statement to determine the displacement field $v_c^{\bullet\bullet}$ of expansion (10.4). The associated differential equations are:

$$\begin{aligned} Av_c^{\bullet\bullet} + C'(u_c) v_c^{\bullet\bullet} &= -a_i a_j C''(u_c) v_c^{\bullet(i)} v_c^{\bullet(j)} - 2\lambda_c^{\bullet} a_i C''(u_c) u_c^{\Diamond} v_c^{\bullet(i)} \\ av_c + c''(u_c) v_c^{\bullet\bullet} &= -a_i a_j c''(u_c) v_c^{\bullet(i)} v_c^{\bullet(j)} - 2\lambda_c^{\bullet} a_i c''(u_c) u_c^{\Diamond} v_c^{\bullet(i)} \end{aligned} , \tag{10.11}$$

where the solution $v_c^{\bullet\bullet}$ has to satisfy also the orthogonality condition (10.5).

If now $\{P''(u_c) v_c^{\bullet(i)} v_c^{\bullet(j)}, v_c^{\bullet(k)}\} = 0$ for all $i,j,k = 1,\ldots,N$, then $\lambda_c^{\bullet} = 0$ and we have symmetric bifurcations. In this case the variational statement $(10.6)_3$ yields:

$$\{P'(u_c) v_c^{\bullet\bullet\bullet} + 3a_i P''(u_c) v_c^{\bullet(i)} v_c^{\bullet\bullet} + a_i a_j a_k P'''(u_c) v_c^{\bullet(i)} v_c^{\bullet(j)} v_c^{\bullet(k)} + $$
$$+ 3\lambda_c^{\bullet\bullet} a_i P''(u_c) u_c^{\Diamond} v_c^{\bullet(i)}, \hat{u}\} = 0 \qquad \forall \hat{u} . \tag{10.12}$$

Setting successively $\hat{u} = v_c^{\cdot(1)}, v_c^{\cdot(2)}, \ldots, v_c^{\cdot(N)}$ and with

$$\{P'(u_c)v_c^{\cdot\cdot\cdot}, v_c^{\cdot(1)}\} = \{P'(u_c)v_c^{\cdot\cdot\cdot}, v_c^{\cdot(2)}\} = \ldots = \{P'(u_c)v_c^{\cdot\cdot\cdot}, v_c^{\cdot(N)}\} = 0 \quad (10.13)$$

(10.12) yields a set of N equations. The load coefficients $\lambda_c^{\cdot\cdot}$ for symmetric bifurcations is obtained as:

$$\lambda_c^{\cdot\cdot} = -\frac{a_i a_j \{P''(u_c)v_c^{\cdot(i)} v_c^{\cdot\cdot}, v_c^{\cdot(j)}\} + \frac{1}{3} a_i a_j a_k a_l \{P'''(u_c)v_c^{\cdot(i)} v_c^{\cdot(j)} v_c^{\cdot(k)}, v_c^{\cdot(l)}\}}{a_i a_j \{P''(u_c)u_c^{\diamond} v_c^{\cdot(i)}, v_c^{\cdot(j)}\}} . \quad (10.14)$$

If in the stationarity condition $(10.6)_2$ we choose $\hat{u} = v_c^{\cdot\cdot}$, it follows for $\lambda_c^{\cdot} = 0$ and with $(A.13)_3$:

$$a_i a_j \{P''(u_c)v_c^{\cdot(i)} v_c^{\cdot(j)}, v_c^{\cdot\cdot}\} = a_i a_j \{P''(u_c)v_c^{\cdot(i)} v_c^{\cdot\cdot}, v_c^{\cdot(j)}\} = -\{P'(u_c)v_c^{\cdot\cdot}, v_c^{\cdot\cdot}\} \quad (10.15)$$

yielding the coefficient $\lambda_c^{\cdot\cdot}$ for symmetric bifurcations in the form:

$$\lambda_c^{\cdot\cdot} = -\frac{a_i a_j a_k a_l \{P'''(u_c)v_c^{\cdot(i)} v_c^{\cdot(j)} v_c^{\cdot(k)}, v_c^{\cdot(l)}\} - 3\{P'(u_c)v_c^{\cdot\cdot}, v_c^{\cdot\cdot}\}}{3 a_i a_j \{P''(u_c)u_c^{\diamond} v_c^{\cdot(i)}, v_c^{\cdot(j)}\}} . \quad (10.16)$$

To define the stability behaviour of the shell structure at bifurcation points, we have to determine the energy increment $\Delta J_p(u_c ; u^b)$ according to (6.6) for all possible paths (10.4). It can be shown that a necessary condition for stability is

$$\{P''(u_c)v_c^{\cdot(i)} v_c^{\cdot(j)}, v_c^{\cdot(k)}\} = 0 \qquad i,j,k = 1,\ldots,N \quad (10.17)$$

or, equivalently, $\lambda_c^{\cdot} = 0$.

A further necessary condition for stability is

$$a_i a_j a_k a_l \{P'''(u_c)v_c^{\cdot(i)} v_c^{\cdot(j)} v_c^{\cdot(k)}, v_c^{\cdot(l)}\} - 3\{P'(u_c)v_c^{\cdot\cdot}, v_c^{\cdot\cdot}\} > 0 \quad (10.18)$$

for which the increment $\Delta J_p(u_c ; u^b)$ is positive. The stability conditions (10.17) and (10.18) correspond to results of Koiter given in [10] for an unspecified functional.

APPENDIX

The governing shell operator $P(u)$ according to (5.6) is defined by:

$$P(u) = \begin{bmatrix} Au + C(u) & \text{in } M \\ au + c(u) & \text{on } C_u \\ au + c(u) & \text{on } C_f \end{bmatrix} , \quad \text{(A.1)}$$

where the first row of (A.1) describes the left side of the nonlinear equilibrium equations (2.8) and the second and third row the left side of the nonlinear static boundary conditions (2.14).

To determine the differential of $P(u)$ at a point $\bar{u}$ in the direction of $\hat{u}$, where $\hat{u}$ is a small deformation superimposed upon the deformation $\bar{u}$, let us consider the expression:

$$\lim_{t \to 0} \frac{P(\bar{u} + t\hat{u}) - P(\bar{u})}{t} = P'(\bar{u} ; \hat{u}) \quad . \quad \text{(A.2)}$$

If this limes is linear in $\hat{u}$ with $P'(\bar{u} ; \hat{u}) = P'(\bar{u})\hat{u}$ then expression (A.2) is called the Gâteaux differential of the nonlinear operator $P(\bar{u})$ with the first Gâteaux derivative $P'(\bar{u})$.

Let $\hat{u}$ and $\check{u}$ be two different displacement fields superimposed upon $\bar{u}$, the second differential is defined by

$$\lim_{t \to 0} \frac{P'(\bar{u} + \check{u} ; \hat{u}) - P'(\bar{u} ; \hat{u})}{t} = P''(\bar{u})\check{u}\,\hat{u} \quad . \quad \text{(A.3)}$$

The third differential has to be calculated analgously. All higher order differentials disappear identically.

With definition (A.1) the differentials of $P(u)$ are the row operators:

$$P'(\bar{u})\hat{u} = \begin{bmatrix} A\hat{u} + C'(\bar{u})\hat{u} & \text{in } M \\ a\hat{u} + c'(\bar{u})\hat{u} & \text{on } C_u \\ a\hat{u} + c'(\bar{u})\hat{u} & \text{on } C_f \end{bmatrix} ; \quad P''(\bar{u})\check{u}\hat{u} = \begin{bmatrix} C''(\bar{u})\check{u}\hat{u} & \text{in } M \\ c''(\bar{u})\check{u}\hat{u} & \text{on } C_u \\ c''(\bar{u})\check{u}\hat{u} & \text{on } C_f \end{bmatrix} ;$$

$$P''' \check{u}\check{u}\hat{u} = \begin{bmatrix} C''' \check{u}\check{u}\hat{u} & \text{in } M \\ c''' \check{u}\check{u}\hat{u} & \text{on } C_u \\ c''' \check{u}\check{u}\hat{u} & \text{on } C_f \end{bmatrix} \tag{A.4}$$

With the definitions (A.2) and (A.3) the differentials of the nonlinear operator $C(u)$ according to (5.3) can be calculated. Using the following differentials of the stress resultant tensor (5.1) and (5.2)

$$\begin{aligned} N^{\alpha\beta\prime}(\bar{u} ; \hat{u}) &= N_{\ell}^{\alpha\beta\prime}(\bar{u} ; \hat{u}) + N_{n}^{\alpha\beta\prime}(\bar{u} ; \hat{u}) \\ N^{\alpha\beta\prime\prime}(\check{u}, \hat{u}) &= N_{n}^{\alpha\beta\prime\prime}(\check{u}, \hat{u}) \end{aligned} \tag{A.5}$$

with

$$\begin{aligned} N_{\ell}^{\alpha\beta\prime}(\bar{u} ; \hat{u}) &= H^{\alpha\beta\lambda\mu}\hat{\theta}_{\lambda\mu} \quad ; \quad N_{\ell}^{\alpha\beta\prime\prime}(\check{u}, \hat{u}) = 0 \\ N_{n}^{\alpha\beta\prime}(\bar{u} ; \hat{u}) &= H^{\alpha\beta\lambda\mu}[\underline{\bar{\varphi}_{\lambda}\hat{\varphi}_{\mu}} + \underline{a_{\lambda\mu}\bar{\varphi}_{3}\hat{\varphi}_{3}} - \underline{\underline{(\hat{\theta}^{\kappa}_{\lambda}\bar{\omega}_{\kappa\mu} + \bar{\theta}^{\kappa}_{\lambda}\hat{\omega}_{\kappa\mu})}}] \\ N_{n}^{\alpha\beta\prime\prime}(\check{u}, \hat{u}) &= H^{\alpha\beta\lambda\mu}[\underline{\check{\varphi}_{\lambda}\hat{\varphi}_{\mu}} + \underline{a_{\lambda\mu}\check{\varphi}_{3}\hat{\varphi}_{3}} - \underline{\underline{(\check{\theta}^{\kappa}_{\lambda}\hat{\omega}_{\kappa\mu} + \hat{\theta}^{\kappa}_{\lambda}\check{\omega}_{\kappa\mu})}}] \end{aligned} \tag{A.6}$$

we obtain:

$$C'(\bar{u})\hat{u} = -\begin{bmatrix} [N_{n}^{\alpha\beta\prime}(\bar{u} ; \hat{u}) - \underline{\tfrac{1}{2}(\bar{\omega}^{\alpha\beta}N_{\lambda}^{\lambda\prime}(\bar{u} ; \hat{u}) + \hat{\omega}^{\alpha\beta}\bar{N}_{\lambda}^{\lambda})} - \\ - \underline{\tfrac{1}{2}(\bar{\omega}^{\alpha\lambda}N_{\lambda}^{\beta\prime}(\bar{u} ; \hat{u}) + \hat{\omega}^{\alpha\lambda}\bar{N}_{\lambda}^{\beta}) - \tfrac{1}{2}(\bar{\omega}^{\beta\lambda}N_{\lambda}^{\alpha\prime}(\bar{u} ; \hat{u}) + \hat{\omega}^{\beta\lambda}\bar{N}_{\lambda}^{\alpha})} + \\ + \underline{\underline{\tfrac{1}{2}(\bar{\theta}^{\alpha\lambda}N_{\lambda}^{\beta\prime}(\bar{u} ; \hat{u}) + \hat{\theta}^{\alpha\lambda}\bar{N}_{\lambda}^{\beta}) - \tfrac{1}{2}(\bar{\theta}^{\beta\lambda}N_{\lambda}^{\alpha\prime}(\bar{u} ; \hat{u}) + \hat{\theta}^{\beta\lambda}\bar{N}_{\lambda}^{\alpha})}}]|_{\beta} - \\ - \underline{b^{\alpha}_{\beta}(\bar{\varphi}_{\lambda}N^{\lambda\beta\prime}(\bar{u} ; \hat{u}) + \hat{\varphi}_{\lambda}\bar{N}^{\lambda\beta})} \\ \underline{[\bar{\varphi}_{\alpha}N^{\alpha\beta\prime}(\bar{u} ; \hat{u}) + \hat{\varphi}_{\alpha}\bar{N}^{\alpha\beta}]|_{\beta}} + b_{\alpha\beta}[N_{n}^{\alpha\beta\prime}(\bar{u};\hat{u}) - \underline{\underline{\bar{\omega}^{\alpha\lambda}N_{\lambda}^{\beta\prime}(\bar{u} ; \hat{u}) - \hat{\omega}^{\alpha\lambda}\bar{N}_{\lambda}^{\beta}}}] \end{bmatrix} \tag{A.7}$$

$$C''(\bar{u})\check{u}\,\hat{u} = -\begin{pmatrix} \begin{aligned} &[N_n^{\alpha\beta''}(\check{u},\hat{u}) - \tfrac{1}{2}\,(\bar{\omega}^{\alpha\beta} N_\lambda^{\lambda''}(\check{u},\hat{u}) + \check{\omega}^{\alpha\beta} N_\lambda^{\lambda'}(\bar{u};\hat{u}) + \hat{\omega}^{\alpha\beta} N_\lambda^{\lambda'}(\bar{u};\check{u})) - \\ &\quad - \tfrac{1}{2}\,(\bar{\omega}^{\alpha\lambda} N_\lambda^{\beta''}(\check{u},\hat{u}) + \check{\omega}^{\alpha\lambda} N_\lambda^{\beta'}(\bar{u};\hat{u}) + \hat{\omega}^{\alpha\lambda} N_\lambda^{\beta'}(\bar{u};\check{u})) - \\ &\quad - \tfrac{1}{2}\,(\bar{\omega}^{\beta\lambda} N_\lambda^{\alpha''}(\check{u},\hat{u}) + \check{\omega}^{\beta\lambda} N_\lambda^{\alpha'}(\bar{u};\hat{u}) + \hat{\omega}^{\beta\lambda} N_\lambda^{\alpha'}(\bar{u};\check{u})) + \\ &\quad + \tfrac{1}{2}\,(\bar{\theta}^{\alpha\lambda} N_\lambda^{\beta''}(\check{u},\hat{u}) + \check{\theta}^{\alpha\lambda} N_\lambda^{\beta'}(\bar{u};\hat{u}) + \hat{\theta}^{\alpha\lambda} N_\lambda^{\beta'}(\bar{u};\check{u})) - \\ &\quad - \tfrac{1}{2}\,(\bar{\theta}^{\beta\lambda} N_\lambda^{\alpha''}(\check{u},\hat{u}) + \check{\theta}^{\beta\lambda} N_\lambda^{\alpha'}(\bar{u};\check{u}) + \hat{\theta}^{\beta\lambda} N_\lambda^{\alpha'}(\bar{u};\check{u})]|_\beta - \\ &\quad - b_\beta^\alpha[\bar{\varphi}_\lambda N^{\lambda\beta''}(\check{u},\hat{u}) + \check{\varphi}_\lambda N^{\lambda\beta'}(\bar{u};\hat{u}) + \hat{\varphi}_\lambda N^{\lambda\beta'}(\bar{u};\check{u})] \\ &[\bar{\varphi}_\alpha N^{\alpha\beta''}(\check{u},\hat{u}) + \check{\varphi}_\alpha N^{\alpha\beta'}(\bar{u};\check{u}) + \hat{\varphi}_\alpha N^{\alpha\beta'}(\bar{u};\check{u})]|_\beta \\ &\quad + b_{\alpha\beta}[N_n^{\alpha\beta''}(\check{u},\hat{u}) - (\bar{\omega}^{\alpha\lambda} N_\lambda^{\beta''}(\check{u},\hat{u}) + \check{\omega}^{\alpha\lambda} N_\lambda^{\beta'}(\bar{u};\hat{u}) + \hat{\omega}^{\alpha\lambda} N_\lambda^{\beta'}(\bar{u};\check{u}))] \end{aligned} \end{pmatrix}$$

(A.8)

$$C'''\check{u}\,\check{u}\,\hat{u} = -\begin{pmatrix} \begin{aligned} &[-\check{\omega}^{\alpha\beta} N_\lambda^{\lambda''}(\check{u},\hat{u}) - \tfrac{1}{2}\,\hat{\omega}^{\alpha\beta} N_\lambda^{\lambda''}(\check{u},\check{u}) - \check{\omega}^{\alpha\lambda} N_\lambda^{\beta''}(\check{u},\hat{u}) - \tfrac{1}{2}\,\hat{\omega}^{\alpha\lambda} N_\lambda^{\beta''}(\check{u},\check{u}) \\ &\quad - \check{\omega}^{\beta\lambda} N_\lambda^{\alpha''}(\check{u},\hat{u}) - \tfrac{1}{2}\,\hat{\omega}^{\beta\lambda} N_\lambda^{\alpha''}(\check{u},\check{u}) + \check{\theta}^{\alpha\lambda} N_\lambda^{\beta''}(\check{u},\hat{u}) + \tfrac{1}{2}\,\hat{\theta}^{\alpha\lambda} N_\lambda^{\beta''}(\check{u},\check{u}) - \\ &\quad - \check{\theta}^{\beta\lambda} N_\lambda^{\alpha''}(\check{u},\hat{u}) - \tfrac{1}{2}\,\hat{\theta}^{\beta\lambda} N_\lambda^{\alpha''}(\check{u},\hat{u})]|_\beta - \\ &\quad - b_\beta^\alpha[2\check{\varphi}_\lambda N^{\lambda\beta''}(\check{u},\hat{u}) + \hat{\varphi}_\lambda N^{\lambda\beta''}(\check{u},\check{u})] \\ &[2\check{\varphi}_\alpha N^{\alpha\beta''}(\check{u},\hat{u}) + \hat{\varphi}_\alpha N^{\alpha\beta''}(\check{u},\check{u})]|_\beta + b_{\alpha\beta}[-2\check{\omega}^{\alpha\lambda} N_\lambda^{\beta''}(\check{u},\hat{u}) - \hat{\omega}^{\alpha\lambda} N_\lambda^{\beta''}(\check{u},\check{u})] \end{aligned} \end{pmatrix}$$

(A.9)

Analogously we calculate the differentials of the nonlinear boundary operator $C(u)$ according to (5.4):

$$C'(\bar{u})\hat{u} = \begin{pmatrix} \begin{aligned} &\nu_\alpha \nu_\beta [N_n^{\alpha\beta'}(\bar{u};\hat{u}) - \hat{\omega}^{\alpha\lambda}\bar{N}_\lambda^\beta - \bar{\omega}^{\alpha\lambda} N_\lambda^{\beta'}(\bar{u};\hat{u})] \\ &t_\alpha \nu_\beta [N_n^{\alpha\beta'}(\bar{u};\hat{u}) - \tfrac{1}{2}\,(\hat{\omega}^{\alpha\beta}\bar{N}_\lambda^\lambda + \bar{\omega}^{\alpha\beta} N_\lambda^{\lambda'}(\bar{u};\hat{u})) - \\ &\quad - \tfrac{1}{2}(\hat{\omega}^{\alpha\lambda}\bar{N}_\lambda^\beta + \bar{\omega}^{\alpha\lambda} N_\lambda^{\beta'}(\bar{u};\hat{u}) + \hat{\omega}^{\beta\lambda}\bar{N}_\lambda^\alpha + \bar{\omega}^{\beta\lambda} N_\lambda^{\alpha'}(\bar{u};\hat{u})) + \\ &\quad + \tfrac{1}{2}(\hat{\theta}^{\alpha\lambda}\bar{N}_\lambda^\beta + \bar{\theta}^{\alpha\lambda} N_\lambda^{\beta'}(\bar{u};\hat{u}) - \hat{\theta}^{\beta\lambda}\bar{N}_\lambda^\alpha - \bar{\theta}^{\beta\lambda} N_\lambda^{\alpha'}(\bar{u};\hat{u}))] \\ &\nu_\beta[\hat{\varphi}_\alpha \bar{N}^{\alpha\beta} + \bar{\varphi}_\alpha N^{\alpha\beta'}(\bar{u};\hat{u})] \\ &0 \\ &0 \end{aligned} \end{pmatrix} \qquad (A.10)$$

$$
c''(\bar{u})\check{u}\,\hat{u} = \left\{ \begin{array}{l}
\nu_\alpha \nu_\beta [N_n^{\alpha\beta''}(\check{u},\hat{u}) - \hat{\omega}^{\alpha\lambda} N_\lambda^{\beta'}(\bar{u};\check{u}) - \check{\omega}^{\alpha\lambda} N_\lambda^{\beta'}(\bar{u};\hat{u}) - \bar{\omega}^{\alpha\lambda} N_\lambda^{\beta''}(\check{u},\hat{u})] \\
t_\alpha \nu_\beta [N_n^{\alpha\beta''}(\check{u},\hat{u}) - \frac{1}{2}(\hat{\omega}^{\alpha\beta} N_\lambda^{\lambda'}(\bar{u};\check{u}) + \check{\omega}^{\alpha\beta} N_\lambda^{\lambda'}(\bar{u};\hat{u}) + \bar{\omega}^{\alpha\beta} N_\lambda^{\lambda''}(\check{u},\hat{u})) - \\
\qquad - \frac{1}{2}(\hat{\omega}^{\alpha\lambda} N_\lambda^{\beta'}(\bar{u};\check{u}) + \check{\omega}^{\alpha\lambda} N_\lambda^{\beta'}(\bar{u};\hat{u}) + \bar{\omega}^{\alpha\lambda} N_\lambda^{\beta''}(\check{u},\hat{u}) + \hat{\omega}^{\beta\lambda} N_\lambda^{\alpha'}(\bar{u};\check{u}) + \\
\qquad + \check{\omega}^{\beta\lambda} N_\lambda^{\alpha'}(\bar{u};\hat{u}) + \bar{\omega}^{\beta\lambda} N_\lambda^{\alpha''}(\check{u},\hat{u})) + \\
\qquad + \frac{1}{2}(\hat{\theta}^{\alpha\lambda} N_\lambda^{\beta'}(\bar{u};\check{u}) + \check{\theta}^{\alpha\lambda} N_\lambda^{\beta'}(\bar{u};\hat{u}) + \bar{\theta}^{\alpha\lambda} N_\lambda^{\beta''}(\check{u},\hat{u}) - \\
\qquad - \hat{\theta}^{\beta\lambda} N_\lambda^{\alpha'}(\bar{u};\check{u}) - \check{\theta}^{\beta\lambda} N_\lambda^{\alpha'}(\bar{u};\hat{u}) - \bar{\theta}^{\beta\lambda} N_\lambda^{\alpha''}(\check{u},\hat{u})] \\
\nu_\beta [\hat{\varphi}_\alpha N^{\alpha\beta'}(\bar{u};\check{u}) + \check{\varphi}_\alpha N^{\alpha\beta'}(\bar{u};\hat{u}) + \bar{\varphi}_\alpha N^{\alpha\beta''}(\check{u},\hat{u})] \\
0 \\
0
\end{array} \right. \tag{A.11}
$$

$$
c'''\,\check{u}\,\check{u}\,\hat{u} = \left\{ \begin{array}{l}
\nu_\alpha \nu_\beta [-\hat{\omega}^{\alpha\lambda} N_\lambda^{\beta''}(\check{u},\check{u}) - 2\check{\omega}^{\alpha\lambda} N_\lambda^{\beta''}(\check{u},\hat{u})] \\
t_\alpha \nu_\beta [-\frac{1}{2}\hat{\omega}^{\alpha\beta} N_\lambda^{\lambda''}(\check{u},\check{u}) - \check{\omega}^{\alpha\beta} N_\lambda^{\lambda''}(\check{u},\hat{u}) - \\
\qquad - \frac{1}{2}\hat{\omega}^{\alpha\lambda} N_\lambda^{\beta''}(\check{u},\check{u}) - \check{\omega}^{\alpha\lambda} N_\lambda^{\beta''}(\check{u},\hat{u}) - \\
\qquad - \frac{1}{2}\hat{\omega}^{\beta\lambda} N_\lambda^{\alpha''}(\check{u},\check{u}) - \check{\omega}^{\beta\lambda} N_\lambda^{\alpha''}(\check{u},\hat{u}) + \\
\qquad + \frac{1}{2}\hat{\theta}^{\alpha\lambda} N_\lambda^{\beta''}(\check{u},\check{u}) + \check{\theta}^{\alpha\lambda} N_\lambda^{\beta''}(\check{u},\hat{u}) - \\
\qquad - \frac{1}{2}\hat{\theta}^{\beta\lambda} N_\lambda^{\alpha''}(\check{u},\check{u}) - \check{\theta}^{\beta\lambda} N_\lambda^{\alpha''}(\check{u},\hat{u})] \\
\nu_\beta [\hat{\varphi}_\alpha N^{\alpha\beta''}(\check{u},\check{u}) + 2\check{\varphi}_\alpha N^{\alpha\beta''}(\check{u},\hat{u})] \\
0 \\
0
\end{array} \right. \tag{A.12}
$$

If the differentials of the total potential energy of the nonlinear elastic shell (6.7) are integrated by part, the following relations can be proved:

$$
\begin{aligned}
J_p'(\bar{u};\hat{u}) &= \{P(\bar{u}) - F, \hat{u}\} \\
J_p''(\bar{u};\check{u}\,\hat{u}) &= \{P'(\bar{u})\check{u}, \hat{u}\} = \{\check{u}, P'(\bar{u})\hat{u}\} \\
J_p'''(\bar{u};\check{u}\,\check{u}\,\hat{u}) &= \{P''(\bar{u})\check{u}\,\check{u}, \hat{u}\} = \{P''(\bar{u})\check{u}\,\hat{u}, \check{u}\} \\
J_p''''(\bar{u};\check{u}\,\check{u}\,\check{u}\,\hat{u}) &= \{P'''\,\check{u}\,\check{u}\,\check{u}, \hat{u}\}
\end{aligned} \tag{A.13}
$$

where the bilinear form $\{.,.\}$ is defined by (5.8).

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