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Zusammenfassung

In der vorliegenden Arbeit wird die Struktur der Grundgleichungen zur Beschreibung ebener Balkendeformationen unter Verwendung verallgemeinerter Dehnungs- und zugeordneter Spannungsmaße untersucht. Eine einfachste, konsistent formulierte geometrisch nichtlineare Balkentheorie bei Auftreten kleiner Dehnungen und finiter Rotationen wird hergeleitet. Führt man zusätzliche Beschränkungen der Balkendeformationen ein, lassen sich konsistente Vereinfachungen der Grundgleichungen angeben. Es wird gezeigt, daß die Formulierung adequater Balkentheorien einen guten Einblick in nichtlineare Theorien dünner Schalen gibt. So können vereinfachte Schalenvarianten hinsichtlich ihrer Konsistenz einem Vorabtest unterzogen werden. Die Leistungsfähigkeit des theoretischen Konzepts wird anhand verschiedener Testbeispiele aufgezeigt.

Summary

The structure of the basic equations for in-plane finite deflections of rods is analysed in terms of generalized strains and work-conjugate stresses. A most simple consistent nonlinear rod theory for small strains and unrestricted rotations is presented. Consistent simplifications of the basic relations are discussed under additional restrictions imposed on the rod deformation. It is shown that the construction of adequate rod theories gives a good insight into the nonlinear theories of elastic shells and the consistency or inconsistency of simplified variants can be enlightened. Theoretical concepts are illustrated by various numerical examples.

CONTENTS

1. INTRODUCTION

In the theory of thin elastic shells the changes of the shell middle surface metric and curvature tensors from the undeformed to the deformed configuration are used as natural strain measures. However, the fundamental considerations of KOITER, JOHN and others (see [1-4]) showed that any other in-surface and bending strain measure differing from the conventional ones by terms of $O(\eta\theta^2)$ and $O(\eta\theta^2/h)$, respectively, are equivalent in the sense of the first approximation of the shell strain energy. Here η is a maximum strain in the shell, h the shell thickness and θ a common parameter for small quantities defined in [1-4]. Based on these results various modified strain measures have been proposed in the shell literature. It is shown in [5,6] that other than the conventional strain measures are more convenient to formulate the basic shell equations in terms of stress resultants (or stress function) and a finite rotation vector, which had been first proposed by REISSNER for axisymmetric deformations of thin shells of revolution [7]. However, general numerical approximation procedures as finite difference and finite element methods are mainly based on shell equations formulated in terms of displacements.

Modified change of curvature tensors for a displacemental formulation of nonlinear shell theories and Lagrangean shell equations have been derived by KOITER [8] and BUDIANSKY [9], whereas the full set of entirely Lagrangean shell equations for small strains and unrestricted rotations was given by PIETRASZKIEWICZ and SZWABOWICZ [10]. However, these equations are very complex and intractable for general numerical approximation procedures. On the other hand there is a wide variety of simplified shell equations, published in the literature, which have been derived under the assumption of additional restrictions (sometimes not explicitly stated) imposed on the magnitude of displacements and/or rotations. We mention only those given in [8,11-23]. It should be pointed out that all nonlinear shell equations, widely used in theory and engineering practice, are valid only in restricted domaines of applicability, whereas these domains are in general not well-defined. Moreover we have shown recently [17,23] that some shell theories formulated with the attempt to extend the range of applicability are not consistent and lead to unrealistic behaviour of the shell structure. These results initiated the investigations of the present report.

Subject of this paper is the formulation of flexible rod theories for highly nonlinear in-plane deformations under assumptions analogous to those of the first approximation shell theory. Two main topics are analysed. First, generalized strain measures are defined, which include as special cases various known strain measures published in the literature for rod boundary value problems. They yield also the modified strain measures of the nonlinear shell theory in their one-dimensional reduced form. Next, work-conjugate stress resultants and stress couples are defined by applying the virtual work principle, and corresponding field equations are derived in a straightforward way. As a result a unified approach to the in-plane rod boundary value problem is obtained, whereas the mathematical structure of the basic equations and the physical meaning of the associated variables are discussed.

The displacemental formulation of rod boundary value problems is studied in more detail. A most simple fully general theory is presented and consistently simplified versions of it are derived, whereas their range of applicability is established. Finally numerical results are presented for various rod boundary value problems.

The mathematical structure of the equations of in-plane deformations of rods is the same as for the cylindrical bending of shells. Therefore theoretical and numerical results of this paper give a comprehensive description of corresponding shell problems as well. Moreover the derivation of simplified rod theories yields a good insight into similar investigations of the nonlinear theory of shells. In particular a partial verification of various simplified nonlinear shell theories is achieved by a reduction to their one-dimensional form.

There is a vast number of publications on the deformation of rods. For historical review and bibliography the reader is referred to [24,25]. Plane deformations of rods (including material non-linearity) within the frame of the Kirchhoff hypothesis or equivalent assumptions have been considered in [26,27]. Theories of rods with a richer structure (including shear deformation and other effects) and the qualitative behaviour of solutions have been subject of many publications, e.g. [25,28,29]. In the aforementioned papers the basic equations are formulated in terms of static and deformation variables. Less attention has been devoted to a displacemental formulation of rod theories, in particular for the case of unrestricted deflections and/or rotations. Various relations are given in [30-32], however, a full set of equations in terms of displacements with proper boundary conditions has been derived only for the special case of straight beams [33,34].

2. GEOMETRY OF DEFORMATION

We shall consider the smooth in-plane deformation of a flexible rod or what is equivalent the cylindrical deformation of a shell. We assume that cross-sections that are normal to a reference line in the initial configuration of the rod remain planar, normal to the deformed reference line and suffer no strains in their planes (Kirchhoff's hypothesis [24]). However, at this stage of the analysis no restrictions as to the magnitude of strains and displacements (rotations) are imposed. In other words as a model for the rod we take a curved extensional elastica [27].

Within this model the configuration of the rod is determined by a single vector function of a material coordinate s, $s \in [s_1, s_2]$. To have all quantities in physical components we assume s to be the arc length in the initial configuration $\mathcal{M}$ of the reference line with the position vector $\underset{\sim}{r} = \underset{\sim}{r}(s)$. Thus the unit tangent vector and unit normal vector at any point $M \in \mathcal{M}$ are (Fig. 1)

$$\underset{\sim}{e}(s) = \underset{\sim}{r}' = \cos\Phi \underset{\sim}{i} + \sin\Phi \underset{\sim}{j} \ , \quad \underset{\sim}{n}(s) = \underset{\sim}{k} \times \underset{\sim}{e} = -\sin\Phi \underset{\sim}{i} + \cos\Phi \underset{\sim}{j} \ , \tag{2.1}$$

where $\Phi(s)$ denotes the tangent angle to $\mathcal{M}$ and the prime indicates differentiation with respect to the arc length s. By (2.1) the curvature of $\mathcal{M}$ is

$$\sigma(s) = \underset{\sim}{r}'' \cdot \underset{\sim}{n} = \Phi' \ . \tag{2.2}$$

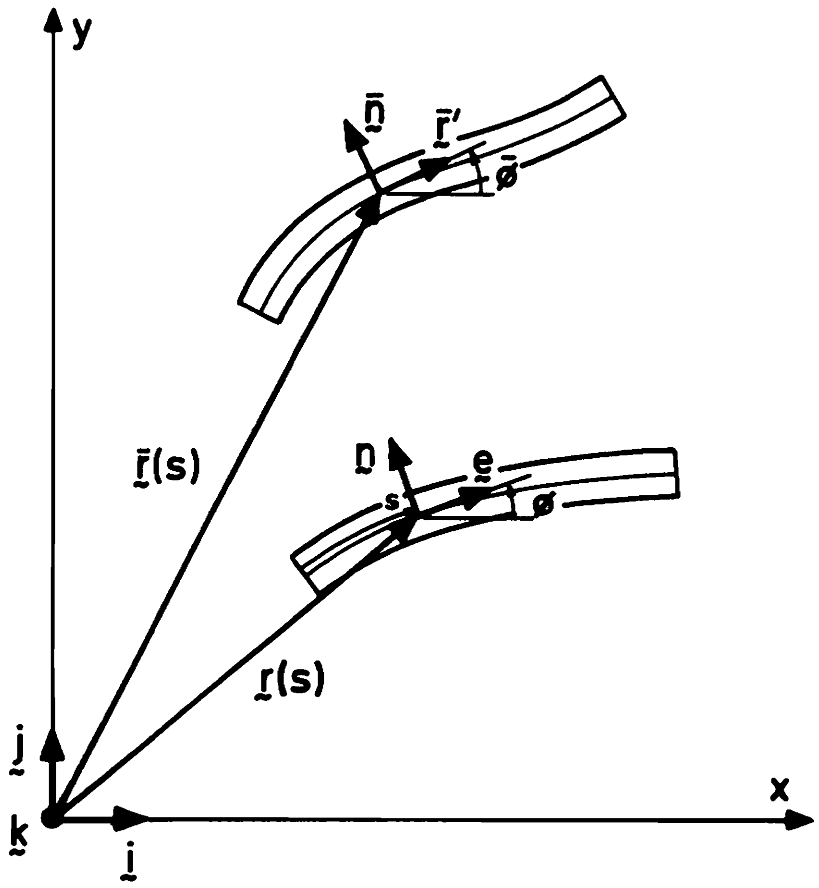

Fig. 1. Geometry of deformation of the rod reference line

Considering the deformation of the rod we take as a rule that all quantities associated to the deformed configuration are distinguished by an additional bar, e.g. the position vector of the deformed reference line $\bar{M}$ is denoted by $\bar{\underset{\sim}{r}} = \bar{\underset{\sim}{r}}(s)$. Obviously $\bar{\underset{\sim}{r}}'$ is the tangent vector to $\bar{M}$ but not necessarily a unit vector, which will be denoted by $\bar{\underset{\sim}{e}}$. Let us define

$$\lambda(s) = d\bar{s}/ds \ , \quad 0 < \lambda < +\infty \ , \tag{2.3}$$

with $d\bar{s} = |\bar{\underset{\sim}{r}}'|$. Then for $\bar{M}$ the following relations hold

$$\begin{aligned} &\bar{\underset{\sim}{r}}'(s) = \lambda(\cos\omega \underset{\sim}{e} + \sin\omega \underset{\sim}{n}) \ , \quad \bar{\underset{\sim}{n}}(s) = \underset{\sim}{k} \times \bar{\underset{\sim}{e}} = -\sin\omega \underset{\sim}{e} + \cos\omega \underset{\sim}{n} \ , \\ &\bar{\sigma} = \lambda^{-2} \bar{\underset{\sim}{r}}'' \cdot \bar{\underset{\sim}{n}} = \lambda^{-1} \bar{\Phi}' \ , \end{aligned} \tag{2.4}$$

where

$$\omega(s) = \bar{\Phi}(s) - \Phi(s) \ . \tag{2.5}$$

The displacement field of the reference line is given by

$$\underset{\sim}{u}(s) = \bar{\underset{\sim}{r}}(s) - \underset{\sim}{r}(s) \ . \tag{2.6}$$

It follows that the local deformation of the rod determined by a mapping of $\{\underset{\sim}{e},\underset{\sim}{n}\}$ into $\{\bar{\underset{\sim}{r}}',\bar{\underset{\sim}{n}}\}$ consists of a parallel translation given by (2.6) and a rigid finite rotation (2.5) followed by a stretch (2.3) of the reference line.

For later use it is desirable to express the relations (2.3-4) in terms of the displacement field

$$\bar{\underset{\sim}{r}}' = \underset{\sim}{e} + \underset{\sim}{u}' \ , \quad \bar{\underset{\sim}{n}} = \lambda^{-1}(\underset{\sim}{n} + \underset{\sim}{k} \times \underset{\sim}{u}') \ . \tag{2.7}$$

As an immediate consequence one gets

$$\begin{aligned} \lambda^2 &= 1 + 2\underset{\sim}{e}\cdot\underset{\sim}{u}' + \underset{\sim}{u}'\cdot\underset{\sim}{u}' \ , \\ \bar{\sigma} &= \lambda^{-3}\{\underset{\sim}{n}\cdot\underset{\sim}{u}'' + (\underset{\sim}{k} \times \underset{\sim}{u}')\cdot\underset{\sim}{u}'' + \sigma[1 + \underset{\sim}{n}\cdot(\underset{\sim}{k} \times \underset{\sim}{u}')]\} \ . \end{aligned} \tag{2.8}$$

The second derivative of the displacement field may be expressed in the form

$$\underset{\sim}{u}'' = (\underset{\sim}{e}\cdot\underset{\sim}{u}')'\underset{\sim}{e} + (\underset{\sim}{n}\cdot\underset{\sim}{u}')'\underset{\sim}{n} + \underset{\sim}{k} \times \underset{\sim}{u}' \ . \tag{2.9}$$

Substituting (2.9) into $(2.8)_2$ the curvature of the deformed reference line is obtained as

$$\bar{\sigma} = \lambda^{-3}[(\underset{\sim}{n}\cdot\underset{\sim}{u}')'(1 + \underset{\sim}{e}\cdot\underset{\sim}{u}') - (\underset{\sim}{e}\cdot\underset{\sim}{u}')'(\underset{\sim}{n}\cdot\underset{\sim}{u}') + \lambda^2\sigma] \ . \tag{2.10}$$

The derivatives of the displacement field can be expressed with the help of (2.7) and $(2.4)_1$ in terms of the stretch and the angle of rotation as follows

$$\begin{aligned} \underset{\sim}{u}' &= (\lambda\cos\omega - 1)\underset{\sim}{e} + \lambda\sin\omega\underset{\sim}{n} \ , \\ \underset{\sim}{u}'' &= (\lambda'\cos\omega - \lambda\omega'\sin\omega)\underset{\sim}{e} + (\lambda'\sin\omega + \lambda\omega'\cos\omega)\underset{\sim}{n} \ . \end{aligned} \tag{2.11}$$

Additional relations may now be derived using component forms of the displacement field with respect to the initial unit base $\{\underset{\sim}{e},\underset{\sim}{n}\}$ and to the unit base $\{\underset{\sim}{i},\underset{\sim}{j}\}$ of the global coordinate system, respectively.

3. STRAIN MEASURES

The strains of the rod reference line are uniquely determined by the stretch λ and the pure change of curvature $\hat{\varkappa} = -\bar{\sigma} + \sigma$. However, these quantities are by no means the only possible strain measures and even not the most suitable ones for the formulation of the general rod boundary value problem. It is apparent that an appropriate choice of the strain measures is closely related to the problem under consideration. In the general case of finite strains this problem is fairly open since, as it has been pointed out by LIBAI and SIMMONDS [35], the conventional strains and changes of curvature are no longer "natural" variables for the description of the deformation. In what follows we shall consider a wide class of strain measures defined by the generalized formulas

$$\tilde{\gamma} = (\lambda^{2k} - 1)/2k \ , \quad \tilde{\varkappa} = - \lambda^{2l_1}\bar{\sigma} + \sigma(1 + l_2\gamma) \ , \tag{3.1}$$

where k, l_1, l_2 are small dimensionless constants. The strains given by (3.1) fulfill basic requirements imposed on strain measures (HILL [36]). In particular they vanish in the initial configuration and for rigid body deformations, i.e. when $\lambda = 1$ and $\bar{\sigma} = \sigma$. In the case of an inextensional deformation ($\lambda = 1$) the definitions (3.1) lead to $\tilde{\gamma} = 0$ and $\tilde{\varkappa} = -\bar{\sigma} + \sigma$ for an arbitrary choice of constants k, l_1 and l_2. Furthermore we shall show later that the strain measures (3.1) become identical with the classical ones in the case of small deformation.

The definition $(3.1)_1$ is the one-dimensional counterpart to HILL's generalized strain tensor [36]. In particular for k = 1 it corresponds to the GREEN strain γ and for k = 1/2 to the nominal ("engineering") strain ε. Definition $(3.1)_2$ contains various flexural strains used in the theory of rods as well as various change of curvature measures used in the shell theory in their reduced one-dimensional form (Tab.1). We shall show in chapter 6 that for small strains all particular cases of the generalized change of curvature measure $(3.1)_2$ are equivalent in the sense of the first approximation to the strain energy function. The choice of one of them is therefore a matter of simplicity of the corresponding field equations. This aspect is studied throughout the paper. It is interesting to note that in the theory of large strains [37] other than the conventional change of curvature measure turn to be more convenient. In this case, however, various flexural measures are no longer equivalent, what may be illustrated by an elementary example of the uniform deformation of a ring (Fig.

k	l_1	l_2	particular form of γ	particular form of $\varkappa$	rod theory	shell theory
1	1	0	$\gamma = (\lambda^2 - 1)/2$	$\varkappa = -\lambda^2\bar{\sigma} + \sigma$		conventional def. [8,19,30]
1	1	1	"	$\rho = -\lambda^2\bar{\sigma} + \sigma(1 + \gamma)$		Koiter [4,8]
1	3/2	0	"	$\hat{\rho} = -\lambda^3\bar{\sigma} + \sigma$		Koiter [8], Simmonds [37]
1	3/2	1	"	$\chi = -\lambda^3\bar{\sigma} + \sigma(1 + \gamma)$		Pietraszkiewicz, Szwabowicz [10]
1	3/2	2	"	$K = -\lambda^3\bar{\sigma} + \sigma(1 + 2\gamma)$	Epstein, Murray [33]*)	Budiansky [9]
1	0	0	"	$\hat{\varkappa} = -\bar{\sigma} + \sigma$		Züricher, Schumann [38]
1/2	1/2	0	$\varepsilon = \lambda - 1$	$\mu = -\lambda\bar{\sigma} + \sigma$	conventional def. [24,27]	Reissner [7]**), Atluri [6]
						Simmonds, Danielson [5]
1/2	0	0	"	$\hat{\varkappa} = -\bar{\sigma} + \sigma$	Tadjbakhsh [26]	

*) theory of straight beams, **) theory of axisymmetric shells of revolution

Table 1. Particular form of generalized strain measures

2).

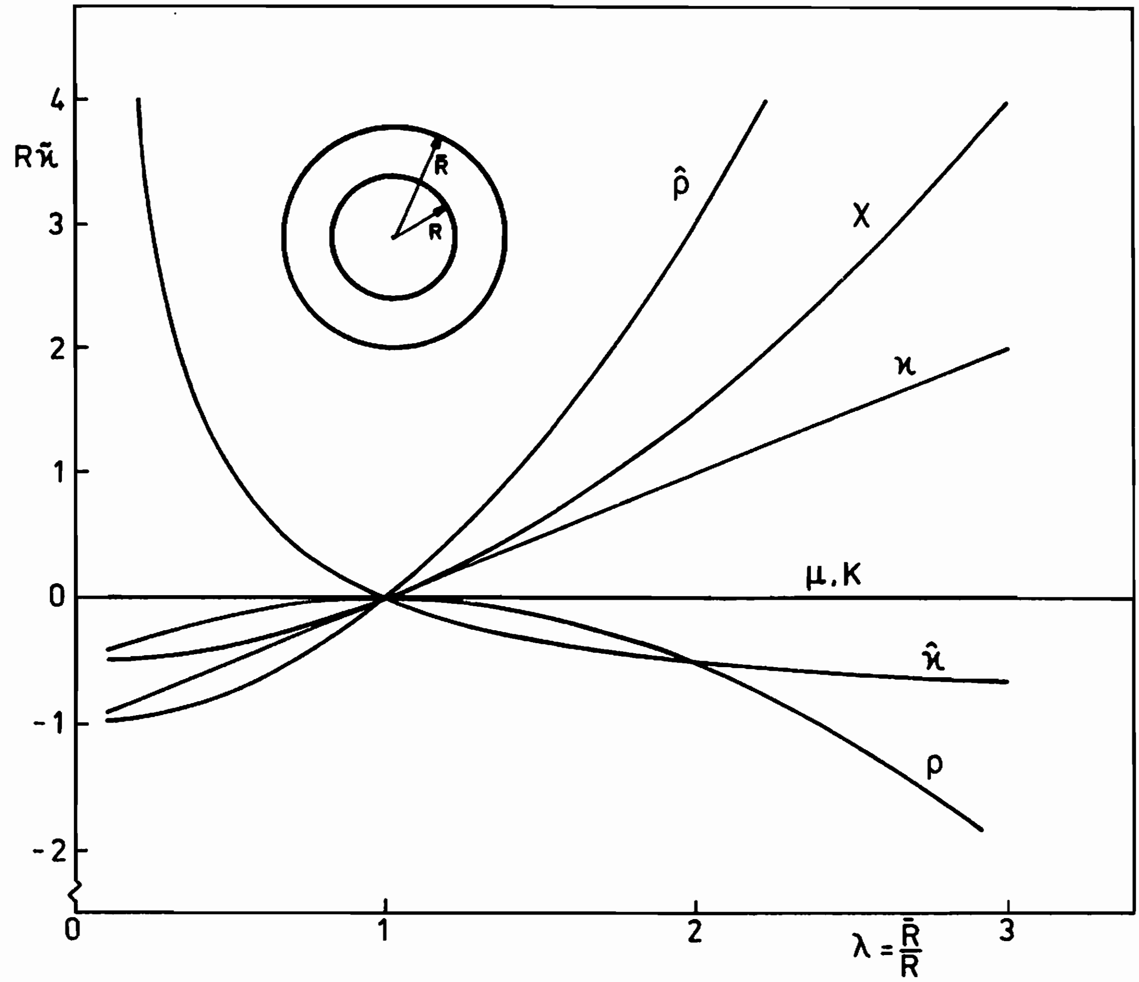

Fig. 2. Uniform deformation of a ring: Comparison of various measures of change of curvature

The definitions (3.1) of strain measures may still be generalized. According to HILL [36] as the extensional strain measure can be taken any smooth monotone function $f_\varepsilon(\lambda)$ of the stretch λ such that $f_\varepsilon(1) = 0$ and $\partial f_\varepsilon/\partial\lambda|_1 = 1$. The definition $(3.1)_1$ fulfills these requirements. In analogy, as the general measure of flexural strains can be taken any smooth function $f_\varkappa(\bar{\sigma},\sigma,\lambda)$ such that $f_\varkappa(\sigma,\sigma,1) = 0$ and $f_\varkappa(\bar{\sigma},\sigma,1) = \pm(\bar{\sigma} - \sigma)$. However, in order to be physically acceptable, some additional conditions like monotony conditions should be imposed on a function $f_\varkappa$.

4. CONJUGATE STRESSES. EQUATIONS OF EQUILIBRIUM

Although the equilibrium equations for rods may be obtained in different ways, their structure, as it was pointed out by ANTMAN [25], is of exactly the same form. Only the mechanical variables may have different physical meaning.

We define a generalized stress resultant $\tilde{N}$ and a generalized stress couple $\tilde{M}$ as coefficients in the invariant expression of the internal virtual work

$$\delta W_i = \int_{s_1}^{s_2} (\tilde{N}\delta\tilde{\gamma} + \tilde{M}\delta\tilde{\varkappa})ds \ , \tag{4.1}$$

where $\delta\tilde{\gamma}$, $\delta\tilde{\varkappa}$ denote the virtual changes of the strain measures (3.1). Since the function $\underset{\sim}{\bar{r}}(s)$ determines the deformation of the rod reference line the virtual changes of the strain measures may be expressed in terms of virtual displacements $\delta\underset{\sim}{\bar{r}}$. Indeed, from (2.4) we have

$$\delta(\lambda^2) = 2\underset{\sim}{\bar{r}}'\cdot\delta\underset{\sim}{\bar{r}}' \ , \qquad \delta(\lambda^2\bar{\sigma}) = \underset{\sim}{\bar{n}}\cdot\delta\underset{\sim}{\bar{r}}'' + \underset{\sim}{\bar{r}}''\cdot\delta\underset{\sim}{\bar{n}} \ . \tag{4.2}$$

Furthermore, the orthogonality conditions of the base vectors $\underset{\sim}{\bar{n}}\cdot\underset{\sim}{\bar{r}}' = 0$, $\underset{\sim}{\bar{n}}\cdot\underset{\sim}{\bar{n}} = 1$ imply the identities

$$\underset{\sim}{\bar{r}}'\cdot\delta\underset{\sim}{\bar{n}} = -\underset{\sim}{\bar{n}}\cdot\delta\underset{\sim}{\bar{r}}' \ , \qquad \underset{\sim}{\bar{n}}\cdot\delta\underset{\sim}{\bar{n}} = 0 \ , \tag{4.3}$$

and by this

$$\underset{\sim}{\bar{r}}''\cdot\delta\underset{\sim}{\bar{n}} = -\lambda^{-1}\lambda'\underset{\sim}{\bar{n}}\cdot\delta\underset{\sim}{\bar{r}}' \ . \tag{4.4}$$

Using (4.2) and (4.4) the virtual changes of the strain measures (3.1) can be expressed in the form

$$\begin{aligned} \delta\tilde{\gamma} &= \lambda^{2(k-1)}\underset{\sim}{\bar{r}}'\cdot\delta\underset{\sim}{\bar{r}}' , \\ \delta\tilde{\varkappa} &= -\{[2(l_1 - 1)\lambda^{2(l_1-1)}\bar{\sigma} - l_2\lambda^{2(k-1)}\sigma]\underset{\sim}{\bar{r}}' - \lambda^{-1}\lambda'\lambda^{2(l_1-1)}\underset{\sim}{\bar{n}}\}\cdot\delta\underset{\sim}{\bar{r}}' \\ &\quad - \lambda^{2(l_1-1)}\underset{\sim}{\bar{n}}\cdot\delta\underset{\sim}{\bar{r}}'' \ . \end{aligned} \tag{4.5}$$

Inserting (4.5) into (4.1) and performing integration by parts the internal virtual work reads

$$\delta W_i = -\int_{s_1}^{s_2} \tilde{\underset{\sim}{T}}'\cdot\delta\bar{\underset{\sim}{r}}ds + [\tilde{\underset{\sim}{T}}\cdot\delta\bar{\underset{\sim}{r}} - (\bar{\underset{\sim}{n}} \times \tilde{\underset{\sim}{M}})\cdot\delta\bar{\underset{\sim}{n}}]_{s_1}^{s_2} , \tag{4.6}$$

where the stress resultant vector $\tilde{\underset{\sim}{T}}$ and stress couple vector $\tilde{\underset{\sim}{M}}$ are given by

$$\begin{aligned}\tilde{\underset{\sim}{T}} = \{&\lambda^{2(k-1)}\tilde{N} - [(2l_1 - 1)\lambda^{2(l_1-1)}\bar{\sigma} - l_2\lambda^{2(k-1)}\sigma]\tilde{M}\}\bar{\underset{\sim}{r}}' + \\ &+ \lambda^{-1}(\lambda\lambda^{2(l_1-1)}\tilde{M})'\bar{\underset{\sim}{n}} ,\end{aligned} \tag{4.7}$$

$$\tilde{\underset{\sim}{M}} = -\lambda\lambda^{2(l_1-1)}\tilde{M}\underset{\sim}{k} . \tag{4.8}$$

We assume furthermore that the loads acting on the rod upper and lower surfaces are reduced to the statically equivalent load acting on the reference line with the intensity $\underset{\sim}{p}(s)$ per unit length of its initial configuration. Then the conditions of equilibrium of the rod can be expressed in the weak form

$$\delta W_i - \int_{s_1}^{s_2} \underset{\sim}{p}\cdot\delta\bar{\underset{\sim}{r}}ds - [\delta W_e]_{s_1}^{s_2} = 0 , \tag{4.9}$$

which may be identified as a one-dimensional principle of virtual work. The last term in (4.9) represents the virtual work of the loads acting at the rod end cross-sections and will be specified in the next chapter.

Whenever vector fields $\tilde{\underset{\sim}{T}}$ and $\underset{\sim}{p}$ are continuous functions, (4.9) together with (4.6) yield the following local equations of equilibrium

$$\tilde{\underset{\sim}{T}}' + \underset{\sim}{p} = 0 , \tag{4.10}$$

where the stress resultant vector $\tilde{\underset{\sim}{T}}$ is defined by (4.7). To establish a physical interpretation of the mechanical variables let us recall that the true stress resultant and stress couple vectors are defined by (LOVE [24])

$$\bar{\underset{\sim}{N}} = \iint_{\bar{A}} \bar{\underset{\sim}{t}}d\bar{A} = \bar{N}\bar{\underset{\sim}{e}} + \bar{Q}\bar{\underset{\sim}{n}} , \quad \bar{\underset{\sim}{M}} = \iint_{\bar{A}} \bar{\xi}\bar{\underset{\sim}{n}} \times \bar{\underset{\sim}{t}}d\bar{A} = -\bar{M}\underset{\sim}{k} , \tag{4.11}$$

where $\bar{\underset{\sim}{t}}$ denotes the true stress vector at the rod cross-section in the deformed configuration obtained from the Cauchy stress tensor $\bar{\underset{\sim}{T}}$ by $\bar{\underset{\sim}{t}} = \bar{\underset{\sim}{T}}\bar{\underset{\sim}{e}}$. In (4.11) $\bar{\xi}$ denotes the coordinate along the normal to the deformed reference line $\bar{M}$ and the integration has to be carried out over the deformed rod cross-section.

The equilibrium equations expressing the global balance of force and moment have the form [24,27]

$$\bar{\underset{\sim}{N}}' + \lambda\bar{\underset{\sim}{p}} = 0 \,,$$
$$\bar{\underset{\sim}{M}}' + \bar{\underset{\sim}{r}}' \times \bar{\underset{\sim}{N}} = 0 \,, \tag{4.12}$$

where $\bar{\underset{\sim}{p}}$ denotes the load vector per unit length of $\bar{M}$. From $(4.12)_2$ we obtain the well-known relations $\bar{Q} = \lambda^{-1}\bar{M}'$ and the system of equations (4.12) reduces to

$$(\bar{N}\bar{\underset{\sim}{e}} + \lambda^{-1}\bar{M}'\bar{\underset{\sim}{n}})' + \lambda\bar{\underset{\sim}{p}} = 0 \tag{4.13}$$

Keeping in mind the definitions (4.7-8) a comparison of the equations (4.13) and (4.10) leads to the following identifications

$$\tilde{\underset{\sim}{T}} \equiv \bar{\underset{\sim}{N}} \,, \quad \tilde{\underset{\sim}{M}} \equiv \bar{\underset{\sim}{M}} \,, \quad \underset{\sim}{p} \equiv \lambda\bar{\underset{\sim}{p}} \,. \tag{4.14}$$

In other words the components of the stress resultant vector (4.7) with respect to the unit base vectors in the deformed configuration of the rod are the true axial and the true shearing-forces. Therefore we may write

$$\tilde{\underset{\sim}{T}} = \bar{N}\bar{\underset{\sim}{e}} + \bar{Q}\bar{\underset{\sim}{n}} \,,$$
$$\bar{N} = \lambda\{\lambda^{2(k-1)}\tilde{N} - [(2l_1 - 1)\lambda^{2(l_1-1)}\bar{\sigma} - l_2\lambda^{2(k-1)}\sigma]\tilde{M}\} \,, \tag{4.15}$$
$$\bar{Q} = -\lambda^{-1}(\tilde{\underset{\sim}{M}}\cdot\underset{\sim}{k})' = \lambda^{-1}(\lambda\lambda^{2(l_1-1)}\tilde{M})' \,.$$

Furthermore the component of the stress couple (4.8) along the unit vector $\underset{\sim}{k}$ is the true stress couple

$$\tilde{\underset{\sim}{M}} = -\bar{M}\underset{\sim}{k} \,, \qquad \bar{M} = \lambda\lambda^{2(l_1-1)}\tilde{M} \,. \tag{4.16}$$

The relations $(4.15)_1$ and (4.16) can be inverted leading to the generalized stress resultant $\tilde{N}$ and the stress couple $\tilde{M}$ as a function of the true variables:

$$\tilde{N} = \lambda^{-1}\{\lambda^{2(1-k)}\bar{N} + [(2l_1 - 1)\lambda^{2(1-k)}\bar{\sigma} - l_2\lambda^{2(1-l_1)}\sigma]\bar{M}\} \,,$$
$$\tilde{M} = \lambda^{-1}\lambda^{2(1-l_1)}\,\bar{M} \,. \tag{4.17}$$

These relations show the physical meaning of the generalized stress measures. In correspondence to Tab. 1 we list in Tab. 2 the particular form of the strain measures with associated conjugate stress measures as functions of the true stresses $\bar{N}$ and $\bar{M}$. It follows that the strain measures ε, μ are work-conjugate to the true stress variables $\bar{N}$ and $\bar{M}$.

k	l_1	l_2	strains	conjugate stresses
1	1	0	γ $\varkappa$	$N = \lambda^{-1}(\bar{N} + \bar{\sigma}\bar{M})$ $M = \lambda^{-1}\bar{M}$
1	1	1	γ ρ	$N = \lambda^{-1}([\bar{N} + (\bar{\sigma} - \sigma)\bar{M}]$ $M = \lambda^{-1}\bar{M}$
1	3/2	0	γ $\hat{\rho}$	$N = \lambda^{-1}(\bar{N} + 2\bar{\sigma}\bar{M})$ $M = \lambda^{-2}\bar{M}$
1	3/2	1	γ χ	$N = \lambda^{-1}[\bar{N} + (2\bar{\sigma} - \lambda^{-1}\sigma)\bar{M}]$ $M = \lambda^{-2}\bar{M}$
1	3/2	2	γ K	$N = \lambda^{-1}[\bar{N} + 2(\bar{\sigma} - \lambda^{-1}\sigma)\bar{M}]$ $M = \lambda^{-2}\bar{M}$
1	0	0	γ $\hat{\varkappa}$	$N = \lambda^{-1}(\bar{N} - \bar{\sigma}\bar{M})$ $M = \lambda\bar{M}$
1/2	1/2	0	ε μ	$N = \bar{N}$ $M = \bar{M}$
1/2	0	0	ε $\hat{\varkappa}$	$N = \bar{N} - \lambda\bar{\sigma}\bar{M}$ $M = \bar{M}$

Table 2. Strain measures and work-conjugate stresses

The vector equilibrium equations (4.10) can now be presented in various component forms. Using the relations (4.15-16) we obtain the classical equilibrium equations expressed with respect to the deformed configuration $\bar{M}$ [24,27]

$$
\begin{aligned}
&\bar{N}' - \lambda\bar{\sigma}\bar{Q} + \bar{\underset{\sim}{p}}\cdot\underset{\sim}{\bar{e}} = 0 \ , \\
&\bar{Q}' + \lambda\bar{\sigma}\bar{N} + \bar{\underset{\sim}{p}}\cdot\underset{\sim}{\bar{n}} = 0 \ , \\
&\bar{M}' - \lambda\bar{Q} = 0 \ .
\end{aligned}
\tag{4.18}
$$

Introducing $(4.15)_{2,3}$ and (4.16) these equations may be expressed in terms of the generalized stress and couple resultants.

A different scalar form of the equilibrium equations results from representing the stress resultant vector (4.15) in the global coordinate system {x,y}

$$\tilde{\underset{\sim}{T}} = H\underset{\sim}{i} + V\underset{\sim}{j}$$
$$H = \cos\bar{\Phi}\bar{N} - \sin\bar{\Phi}\bar{Q} \ , \quad V = \sin\bar{\Phi}\bar{N} + \cos\bar{\Phi}\bar{Q} \ . \tag{4.19}$$

Then the equations (4.10) take the form

$$H' + \bar{\underset{\sim}{p}}\cdot\underset{\sim}{i} = 0$$
$$V' + \bar{\underset{\sim}{p}}\cdot\underset{\sim}{j} = 0 \tag{4.20}$$
$$\tilde{M}' + \lambda(\sin\bar{\Phi}H - \cos\bar{\Phi}V) = 0$$

To express the horizontal and vertical components H, V in terms of $\tilde{N}$ and $\tilde{M}$ the relations (4.19) and (4.15) should be applied.

Finally let us consider the form of equilibrium equations expressed entirely with respect to the initial configuration of the rod. With the help of (4.7), (2.7) and (2.10) the stress resultant vector $\tilde{\underset{\sim}{T}}$ can be represented by

$$\tilde{\underset{\sim}{T}} = \tilde{T}\underset{\sim}{e} + \tilde{Q}\underset{\sim}{n} \ ,$$
$$\tilde{T} = (1 + \underset{\sim}{e}\cdot\underset{\sim}{u}')T - (\underset{\sim}{n}\cdot\underset{\sim}{u}')\lambda^{-2}(\lambda\lambda^{2(l_1-1)}\tilde{M})' \ , \tag{4.21}$$
$$\tilde{Q} = (\underset{\sim}{n}\cdot\underset{\sim}{u}')T + (1 + \underset{\sim}{e}\cdot\underset{\sim}{u}')\lambda^{-2}(\lambda\lambda^{2(l_1-1)}\tilde{M})' \ ,$$

where

$$T = \lambda^{2(k-1)}\tilde{N} - \{(2l_1 - 1)\lambda^{2(l_1-5/2)}\cdot$$
$$\cdot[(1 + \underset{\sim}{e}\cdot\underset{\sim}{u}')(\underset{\sim}{n}\cdot\underset{\sim}{u}')' - (\underset{\sim}{n}\cdot\underset{\sim}{u}')(\underset{\sim}{e}\cdot\underset{\sim}{u}')'] + \tag{4.22}$$
$$+ \sigma[(2l_1 - 1)\lambda^{2(l_1-3/2)} - l_2\lambda^{2(k-1)}]\}\tilde{M} \ ,$$

The stretch λ in terms of the displacements is given by $(2.8)_1$. Hence (4.10) reads

$$\tilde{T}' - \sigma\tilde{Q} + \underset{\sim}{p}\cdot\underset{\sim}{e} = 0 \ ,$$
$$\tilde{Q}' + \sigma\tilde{T} + \underset{\sim}{p}\cdot\underset{\sim}{n} = 0 \ , \tag{4.23}$$

and by (4.21-4.22) and $(2.8)_1$ they may be presented entirely in terms of the generalized stress and couple resultants $\tilde{N}$, $\tilde{M}$, the displacements and their derivatives, respectively.

5. BOUNDARY CONDITIONS

The boundary conditions are obtained using the principle of virtual work (4.9). We assume that the loads acting at the end cross-sections of the rod are equivalent to a load vector $\underset{\sim}{f}^*$ per unit area of the end cross-sections in the initial configuration (Fig. 3). In general $\underset{\sim}{f}^*$ is a function of the normal

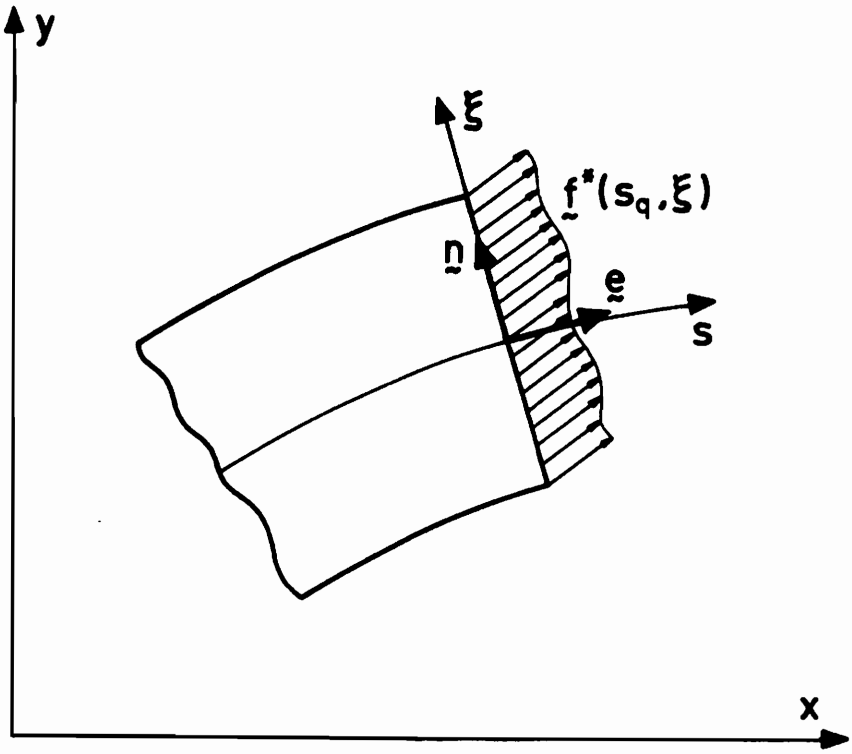

Fig. 3. Loads acting at the rod end cross-sections

coordinate ξ, the displacements and their derivatives, $\underset{\sim}{f}^* = \underset{\sim}{f}^*(\xi, \bar{\underset{\sim}{r}}, \bar{\underset{\sim}{r}}', \ldots)$. Whatever its nature is, the work done in any virtual deformation of the rod is given by

$$[\delta W_e]_{s_1}^{s_2} = \Big[\iint_{A_q} \underset{\sim}{f}^* \cdot (\delta \bar{\underset{\sim}{r}}' + \xi \delta \bar{\underset{\sim}{n}}) dA\Big]_{s_1}^{s_2} = [\underset{\sim}{F}^* \cdot \delta \bar{\underset{\sim}{r}} + \underset{\sim}{H}^* \cdot \delta \bar{\underset{\sim}{n}}]_{s_1}^{s_2} , \qquad (5.1)$$

where the resultant load vector $\underset{\sim}{F}^*$ and the resultant static moment $\underset{\sim}{H}^*$ are defined by

$$\underset{\sim}{F}^*(s_q) = \iint_{A_q} \underset{\sim}{f}^* dA \; , \quad \underset{\sim}{H}^*(s_q) = \iint_{A_q} \xi \underset{\sim}{f}^* dA \; , \quad q = 1,2 \; . \tag{5.2}$$

The static moment $\underset{\sim}{H}^*$ has been introduced in [10] in the frame of the nonlinear shell theory.

In view of (5.1) and (4.6) the principle of virtual work (4.9) yields

$$-\int_{s_1}^{s_2} (\underset{\sim}{\tilde{T}}' + \underset{\sim}{p})\cdot\delta\underset{\sim}{\bar{r}}ds + [(\underset{\sim}{\tilde{T}} - \underset{\sim}{F}^*)\cdot\delta\bar{r} + (\underset{\sim}{\bar{n}} \times \underset{\sim}{\tilde{M}} - \underset{\sim}{H}^*)\cdot\delta\underset{\sim}{\bar{n}}]_{s_1}^{s_2} = 0 \; . \tag{5.3}$$

The vanishing of the first term in (5.3) leads to the equilibrium equations (4.10). The terms in the brackets should vanish at the rod ends and this provides a convenient starting point for the discussion of various possible forms of the boundary conditions. Let us note that the deformation of the rod ends is determined by three kinematical variables, i.e. two components of the position vector $\underset{\sim}{\bar{r}}(s_q)$ and one parameter describing the rotation of the end cross-sections. Depending on the choice of this rotational parameter we arrive at various forms of the boundary conditions. In the classical formulation [24,26,27] the angle of rotation is used. From $(2.4)_2$ we have

$$\delta\underset{\sim}{\bar{n}} = -\underset{\sim}{\bar{n}} \times \delta\underset{\sim}{\omega} \; , \quad \underset{\sim}{\omega} = \omega\underset{\sim}{k} \; . \tag{5.4}$$

Hence the terms in brackets in (5.3) can be transformed further

$$[(\underset{\sim}{\tilde{T}} - \underset{\sim}{F}^*)\cdot\delta\underset{\sim}{\bar{r}} + (\underset{\sim}{\tilde{M}} - \underset{\sim}{M}^*)\cdot\delta\underset{\sim}{\omega}]_{s_1}^{s_2} \; , \tag{5.5}$$

where the resultant load couple is defined by

$$\underset{\sim}{M}^*(s_q) = \underset{\sim}{\bar{n}} \times \underset{\sim}{H}^* = \iint_{A_q} \xi\underset{\sim}{\bar{n}} \times \underset{\sim}{f}^* dA = -M^*\underset{\sim}{k} \; . \tag{5.6}$$

From (5.5) we derive the following form of the boundary conditions

$$\left.\begin{array}{lll} \tilde{\mathbf{T}} = \mathbf{F}^* & \text{or} & \bar{\mathbf{r}} = \bar{\mathbf{r}}^* \\ \bar{M} = M^* & \text{or} & \omega = \omega^* \end{array}\right\} \quad \text{at } s = s_1, s_2 \tag{5.7}$$

where an asterisk indicates prescribed quantities at the rod ends. Various component forms of boundary conditions follow from (5.7) and (4.15-16),(4.19) or (4.21). It should be pointed out that the geometric boundary condition $(5.7)_2$, being linear in the angle of rotation, becomes nonlinear with respect to displacements and their derivatives. Indeed, by $(2.11)_1$ and $(2.8)_1$ we have

$$\omega(\mathbf{u}) = \arcsin \frac{\mathbf{n}\cdot\mathbf{u}'}{\sqrt{(1 + \mathbf{e}\cdot\mathbf{u}')^2 + (\mathbf{n}\cdot\mathbf{u}')^2}} . \tag{5.8}$$

A form of the boundary conditions alternative to (5.7) may be derived in a way analogous to that of the general shell theory [10]. Using the decomposition $\bar{\mathbf{n}}(s_q) = n_\nu\mathbf{e} + n\mathbf{n}$, the identity $\bar{\mathbf{n}}\cdot\bar{\mathbf{n}} = 1$ yields

$$n(s_q) = \pm\sqrt{1 + n_\nu^2} . \tag{5.9}$$

Thus the only independent geometric end parameters are the two components of the displacement vector $\mathbf{u}(s_q) = \bar{\mathbf{r}} - \mathbf{r}$ and the component n_ν describing the rotation of the end cross-sections.

Applying the relation

$$\delta\bar{\mathbf{n}}(s_q) = D^{-1}\bar{\mathbf{r}}'\delta n_\nu , \quad D = \mathbf{e}\cdot\bar{\mathbf{r}}' = 1 + \mathbf{e}\cdot\mathbf{u}' , \tag{5.10}$$

the boundary term in (5.3) may be further transformed into

$$[(\tilde{\mathbf{T}} - \mathbf{F}^*)\cdot\delta\mathbf{u} + (D^{-1}\lambda^{21}\tilde{M} - H^*)\delta n_\nu]_{s_1}^{s_2} , \tag{5.11}$$

where

$$H^*(s_q) = D^{-1}\bar{\mathbf{r}}'\cdot\mathbf{H}^* . \tag{5.12}$$

Thus the virtual work principle leads to the following boundary conditions

$$\left.\begin{array}{lll} \tilde{\mathbf{T}} = \mathbf{F}^* & \text{or} & \mathbf{u} = \mathbf{u}^* \\ D^{-1}\lambda^{21}\tilde{M} = H^* & \text{or} & n_\nu = n_\nu^* \end{array}\right\} \quad \text{at } s = s_1,\ s_2 , \tag{5.13}$$

where the corresponding component form follows from (4.21). The rotational boundary parameter n_ν expressed in terms of displacements by using $(2.7)_2$ and $(2.8)_1$ is obtained as

$$n_\nu(\underset{\sim}{u}) = - \frac{\underset{\sim}{n}\cdot\underset{\sim}{u}'}{\sqrt{(1 + \underset{\sim}{e}\cdot\underset{\sim}{u}')^2 + (\underset{\sim}{n}\cdot\underset{\sim}{u}')^2}} . \tag{5.14}$$

On the other hand it follows from $(2.4)_2$ that $n_\nu = -\sin\omega$. Thus, the boundary conditions in the form (5.13) and (5.7) are equivalent and may be transformed to each other.

6. CONSTITUTIVE EQUATIONS

To complete the set of equations for the rod boundary-value problem the equilibrium equations and corresponding boundary conditions, formulated in previous chapters, have to be supplemented by appropriate one-dimensional constitutive equations. For hyperelastic rods, the general form of these equations has been obtained by Tadjbakhish [26]. Alternatively the one-dimensional constitutive equations can be derived by specializing the three-dimensional constitutive laws under Kirchhoff constraints [27].

Although our subsequent considerations will be mainly restricted to the theory of small strains it is of some interest to present one-dimensional constitutive equations for finite strains. We assume that the material of the rod is hyperelastic, homogeneous and isotropic. Then the strain energy is a function $W = W(\lambda_1,\lambda_2,\lambda_3)$ of the principle stretches λ_i, $i = 1, 2, 3$. The constitutive equations in terms of principle components $\bar{T}^{\langle ii\rangle}$ of the Cauchy (true) stress tensor are [39]

$$I\bar{T}^{\langle ii\rangle} = \lambda_i \partial W/\partial\lambda_i \;\; , \; I = \lambda_1\lambda_2\lambda_3 \tag{6.1}$$

Here W is the strain energy per unit volume of the initial configuration of the rod. Within the Kirchhoff hypothesis $\lambda_2 = \lambda_3 = 1$ and

$$\lambda_1 = (1 - \xi\sigma)^{-1}[\lambda - 1 + \xi(- \lambda\bar{\sigma} + \sigma)] \tag{6.2}$$

where $\lambda = \lambda_1(0)$ and ξ denote the stretch and the normal coordinate to the reference line, respectively. With (6.2) the natural strain measures of the rod reference line are

$$\varepsilon = \lambda - 1 \;, \quad \mu = -\,\lambda\bar{\sigma} + \sigma \tag{6.3}$$

As a consequence of the Kirchhoff assumptions the strain energy is a function $W(\lambda_1) = W(\varepsilon,\mu)$ of the strains (6.3). Furthermore from (6.2) it follows by differentiation

$$(1 - \xi\sigma)\;\partial W/\partial\varepsilon = \partial W/\partial\lambda_1 \;, \qquad (1 - \xi\sigma)\;\partial W/\partial\mu = \xi\partial W/\partial\lambda_1 \;. \tag{6.4}$$

Recalling next the definitions (4.11) of the true axial force $\bar{N}$ and true resultant couple $\bar{M}$ and applying (6.1), (6.4) we obtain the following form of the one-dimensional constitutive equations for hyperelastic rods

$$\begin{aligned} \bar{N} &= \iint \bar{T}^{<11>} dA = \iint \partial W/\partial\lambda_1 \; dA = \partial\Sigma/\partial\varepsilon \;, \\ \bar{M} &= \iint \bar{T}^{<11>} \xi dA = \iint \partial W/\partial\lambda_1 \; \xi dA = \partial\Sigma/\partial\mu \;, \end{aligned} \tag{6.5}$$

where Σ is the one-dimensional strain-energy function per unit length of the undeformed reference line

$$\Sigma = \Sigma(\varepsilon,\mu) = \iint W(1 - \xi\sigma) dA \tag{6.6}$$

The constitutive equations in the form (6.5) may be obtained by different arguments. We have shown that $\bar{N}$, $\bar{M}$ and ε, μ are work-conjugate stresses and strains (see Tab. 2). Hence, taking variation of (6.6) and comparing it with the internal virtual work expression (4.1) for the constants $k = l_1 = 1/2$, $l_2 = 0$ we arrive at (6.5).

The constitutive equations (6.5) have been obtained in [27] in a slightly different way. Replacing the measure of change of curvature μ by $\hat{\varkappa} = -\,(\bar{\sigma} - \sigma)$, which is a particular case of $(3.1)_2$, it follows

$$\partial\hat{\varkappa}/\partial\varepsilon = -\,(1 + \varepsilon)^{-2}(\mu - \sigma) \;, \qquad \partial\hat{\varkappa}/\partial\mu = (1 + \varepsilon)^{-1} \;, \tag{6.7}$$

and accordingly with the help of (6.5) we may derive an equivalent form of the constitutive equations

$$\bar{N} = \partial\hat{\Sigma}/\partial\varepsilon + (1 + \varepsilon)^{-1}\bar{\sigma}\ \partial\hat{\Sigma}/\partial\hat{\varkappa}\ ,$$
$$\bar{M} = (1 + \varepsilon)^{-1}\ \partial\hat{\Sigma}/\partial\hat{\varkappa}\ , \tag{6.8}$$

where $\hat{\Sigma}(\varepsilon,\ \hat{\varkappa}) = \Sigma\ [\varepsilon,\ \mu(\varepsilon,\hat{\varkappa})]$. These equations first derived in [27] are, however, more complicated, because $\bar{N}$, $\bar{M}$ and ε, $\hat{\varkappa}$ are not work-conjugates. Let $\hat{N}$, $\hat{M}$, denote work-conjugate stresses with strain measures ε, $\hat{\varkappa}$. From (4.15) taking k = 1/2, $l_1 = l_2 = 0$ one gets (Tab. 2)

$$\hat{N} = \bar{N} - \bar{\sigma}\bar{M}\ , \quad \hat{M} = (1 + \varepsilon)\bar{M}\ , \tag{6.9}$$

and by these equations (6.8) reduces to the form identical to (6.5). Formally expressing strains ε, μ by the generalized ones (3.1) and defining the strain energy $\Sigma = \Sigma\ [\varepsilon(\tilde{\gamma}),\mu(\tilde{\gamma},\ \tilde{\varkappa})]$ we obtain the constitutive equations in terms of the generalized variables

$$\tilde{N} = \partial\Sigma/\partial\tilde{\gamma}\ , \quad \tilde{M} = \partial\Sigma/\partial\tilde{\varkappa}\ . \tag{6.10}$$

The explicite form of the one-dimensional stress-strain relations (6.5) or (6.10) can now be derived whenever the strain-energy function W is specified. The equations (6.10) complete the set of equations for the non-linear rod boundary-value problem in terms of the generalized variables.

Further simplifications of the constitutive equations in the case of small strains follow from considerations given above. However the a priori assumed Kirchhoff hypothesis provides no information about the error of the resulting stress-strain relations. Therefore, our subsequent considerations will be based on fundamental estimations achieved within the first approximation shell theory [1-4]. Thus considering the plane deformation of rods as a problem equivalent to the cylindrical bending of shells we can reflect the same aspects of nonlinear shell theories.

We assume that the rod is thin, $h/R \ll 1$, the deformation is smooth, $(h/L)^2 \ll 1$, and strains in the rod space remain small $\eta \ll 1$. Here h denotes the rod thickness and R, L, η are minimum radius of curvature, minimum wave length of deformation pattern and maximum strain, respectively. Under these assumptions it has been proved that the relative error in LOVE's form of the two-dimensional strain energy function is $O(\theta^2)$ at a sufficient distance to the shell boundary [1-4], where the small dimensionless parameter θ is defined by

$$\theta^2 = \max \{(h/L)^2,\ h/R,\ \eta\} \ll 1 \tag{6.11}$$

Hence in the particular case of cylindrical bending of a shell or in-plane deformation of rods the strain energy per unit length of the undeformed reference line is

$$\Sigma = 1/2\ (C_1\gamma^2 + C_2\varkappa^2)\ [1 + O(\theta)^2)]\ . \tag{6.12}$$

Here C_1, C_2 are material constants characterizing the extensional and bending rigidity of the rod and γ, $\varkappa$ are conventional strain measures (Tab. 1) for which the following estimations hold [2]

$$\gamma = O(\eta), \qquad \varkappa = O(\eta/h)\ . \tag{6.13}$$

It follows from (6.12-13) that strain measures differing from the conventional ones by terms of $O(\eta\theta^2)$ and $O(\eta\theta^2/h)$, respectively, are equivalent in the sense of the first approximation to the strain energy function [4]. We shall show now that this is the case for the generalized strains defined by (3.1).

Let us define the maximum extensional strain of the reference line by

$$\varepsilon_m = \sup_{s_1 < s < s_2} |\lambda(s) - 1| < \eta. \tag{6.14}$$

Since $\varepsilon_m < 1$ the following expansion holds

$$\begin{aligned}\lambda^q = (1+\varepsilon)^q &= 1 + q\varepsilon + q(q-1)\varepsilon^2/2! + \dots + \\ &\quad q(q-1)\dots(q-i+1)\varepsilon^i/i! + \dots\end{aligned} \tag{6.15}$$

for any rational number q. With the help of (6.15) the difference between conventional strains and generalized ones (3.1) may be estimated to be

$$\tilde{\gamma} - \gamma = O[(1-k)\varepsilon_m^2]\ ,\quad \tilde{\varkappa} - \varkappa = O[l_1\ \eta\varepsilon_m/h + (l_2\ - l_1)\varepsilon_m/R\ ] \tag{6.16}$$

Consequently the strain measures $\tilde{\gamma}$ and γ are equivalent in the aforementioned sense whenever $|1 - k| = O(\eta\theta^2/\varepsilon_m^2)$, whereas the changes of curvature $\tilde{\varkappa}$ and $\varkappa$ are equivalent if $|l_1| + |l_2 - l_1| = O(\eta/\varepsilon_m)$.

Under the above restrictions upon the constants k, l_1, l_2 the conventional strain measures in the strain energy expression (6.12) may be replaced by the generalized ones without violation of its accuracy

$$\Sigma = 1/2\ (C_1\tilde{\gamma}^2 + C_2\tilde{\varkappa}^2)\ . \tag{6.17}$$

Consequently, the one-dimensional constitutive equations in terms of the generalized variables according to (6.10) are

$$\tilde{N} = C_1\tilde{\gamma}\ , \quad \tilde{M} = C_2\tilde{\varkappa}\ . \tag{6.18}$$

7. STATIONARY PRINCIPLE OF TOTAL POTENTIAL ENERGY

We turn now to the formulation of rod boundary value-problems in terms of displacements. Using relations $(2.8)_1$ and (2.10) the generalized strains (3.1) may be transformed into:

$$\begin{aligned}
\tilde{\gamma} &= \{[1 + 2(\underset{\sim}{e}\cdot\underset{\sim}{u}') + (\underset{\sim}{e}\cdot\underset{\sim}{u}')^2 + (\underset{\sim}{n}\cdot\underset{\sim}{u}')^2]^k - 1\}/(2k)\ ,\\
\tilde{\varkappa} &= -\ [1 + 2(\underset{\sim}{e}\cdot\underset{\sim}{u}') + (\underset{\sim}{e}\cdot\underset{\sim}{u}')^2 + (\underset{\sim}{n}\cdot\underset{\sim}{u}')^2]^{l_1 - 3/2}\cdot\\
&\quad\cdot\{(1 + \underset{\sim}{e}\cdot\underset{\sim}{u}')(\underset{\sim}{n}\cdot\underset{\sim}{u}')' - (\underset{\sim}{n}\cdot\underset{\sim}{u}')(\underset{\sim}{e}\cdot\underset{\sim}{u}')' +\\
&\quad + \sigma[1 + 2(\underset{\sim}{e}\cdot\underset{\sim}{u}') + (\underset{\sim}{e}\cdot\underset{\sim}{u}')^2 + (\underset{\sim}{n}\cdot\underset{\sim}{u}')^2]\} + \sigma(1 + l_2\,\tilde{\gamma})\ ,
\end{aligned} \tag{7.1}$$

Hence, the strain energy of the rod is a functional

$$U(\underset{\sim}{u}) = 1/2\int_{s_1}^{s_2} [C_1\tilde{\gamma}^2(\underset{\sim}{u}) + C_2\tilde{\varkappa}^2(\underset{\sim}{u})]ds\ , \tag{7.2}$$

defined on the space of displacement fields. Let us assume furthermore that the reduced loading acting along the rod reference line is conservative, i.e. there exists a functional of the potential energy $V(\underset{\sim}{u})$ such that

$$\int_{s_1}^{s_2} \underset{\sim}{p}\cdot\delta\underset{\sim}{u}ds = -\ \delta V(\underset{\sim}{u};\delta\underset{\sim}{u})\ . \tag{7.3}$$

Obviously the dead load is conservative and the potential energy is given by

$$V(\underset{\sim}{u}) = -\int_{s_1}^{s_2} \underset{\sim}{p}\cdot\underset{\sim}{u}ds \ , \quad \underset{\sim}{p} = \text{const.} \tag{7.4}$$

Let us consider next the pressure load. By definition [40] the current load vector remains normal to the deformed reference line and maintains the same magnitude per unit length, i.e.

$$\underset{\sim}{p}ds = - q\bar{\underset{\sim}{n}}d\bar{s} \ , \tag{7.5}$$

where the pressure magnitude q is constant during the rod deformation. Noting that $d\bar{s} = \lambda ds$, from (7.5) and $(2.7)_2$ one gets

$$\underset{\sim}{p}(\underset{\sim}{u}) = - q(\underset{\sim}{n} + \underset{\sim}{k} \times \underset{\sim}{u}') \ . \tag{7.6}$$

Furthermore it is easy to verify the following identity

$$2(\underset{\sim}{u}' \times \delta\underset{\sim}{u}) = - \delta(\underset{\sim}{u} \times \underset{\sim}{u}') + (\underset{\sim}{u} \times \delta\underset{\sim}{u})' \ , \tag{7.7}$$

which enables to transform the virtual work done by pressure load into

$$\int_{s_1}^{s_2} \underset{\sim}{p}(\underset{\sim}{u})\cdot\delta\underset{\sim}{u}ds = - \delta V(\underset{\sim}{u},\delta\underset{\sim}{u}) + [q\underset{\sim}{k}/2 \cdot(\underset{\sim}{u} \times \delta\underset{\sim}{u})]_{s_1}^{s_2} \ , \tag{7.8}$$

where

$$V(\underset{\sim}{u}) = \int_{s_1}^{s_2} q[\underset{\sim}{n}\cdot\underset{\sim}{u} - \underset{\sim}{k}\cdot(\underset{\sim}{u} \times \underset{\sim}{u}')/2]ds \ . \tag{7.9}$$

It follows from (7.8) that the pressure load is conservative with a potential energy given by (7.9), if the last term on the right hand side of (7.8) vanishes, i.e.

$$\underset{\sim}{k}\cdot(\underset{\sim}{u} \times \delta\underset{\sim}{u}) = u\delta w - w\delta u = 0 \quad \text{at } s = s_1, s_2 \ . \tag{7.10}$$

This is the case whenever

$$u = \underset{\sim}{e}\cdot\underset{\sim}{u} = \text{const.} \quad \text{or} \quad w = \underset{\sim}{n}\cdot\underset{\sim}{u} = \text{const.} \quad \text{at } s = s_1, s_2 \ . \tag{7.11}$$

Loads acting at the end cross-sections of the rod are conservative if there exists a potential $V_e[\underset{\sim}{u}(s_q)]$ such that

$$[\delta W_e]_{s_1}^{s_2} = - [\delta V_e(\underset{\sim}{u},\delta\underset{\sim}{u})]_{s_1}^{s_2} . \tag{7.12}$$

An example of conservative end loading is $\underset{\sim}{f}^* =$ const., i.e. dead loads (see [10]) with an associated potential V_e given by

$$[V_e(\underset{\sim}{u})]_{s_1}^{s_2} = [\underset{\sim}{F}^*\cdot\underset{\sim}{u} + \underset{\sim}{H}^*\cdot(\bar{\underset{\sim}{n}}(\underset{\sim}{u}) - \underset{\sim}{n})]_{s_1}^{s_2} , \tag{7.13}$$

where $\underset{\sim}{F}^* =$ const. and $\underset{\sim}{H}^* =$ const. by definition (5.2).

Whenever loads acting on the rod are conservative the functional of total potential energy is defined by

$$J(\underset{\sim}{u}) = U(\underset{\sim}{u}) + V(\underset{\sim}{u}) + [V_e(\underset{\sim}{u})]_{s_1}^{s_2} \tag{7.14}$$

Hence applying standard variational procedure it can be proved that the equilibrium equations (4.23) and the static boundary conditions (5.13) are Euler-Lagrange equations of the variational problem

$$\delta J(\underset{\sim}{u},\delta\underset{\sim}{u}) = 0 . \tag{7.15}$$

for all kinematical admissible variations $\delta\underset{\sim}{u}$ of the displacement field.

Using the functional of the total potential energy (7.14) and applying the general procedure developed by STUMPF [41], the corresponding nonlinear stability equations for rods may be derived.

8. CONSISTENT VARIANTS OF ROD THEORIES

The equivalence of different strain measures in the sense of the first approximation to the strain energy is of considerable importance for at least two reasons. First, because essential simplifications of the basic equations may be achieved by an appropriate selection of the strain measures and second, because it enables to distinguish between significant and irrelevant terms in the various simplified relations of the theory of rods. In what follows a detailed exposition of these two aspects is given for the case when the governing equations are formulated in terms of the displacements in a fully Lagrangean description.

Using the decomposition of the displacement field $\underset{\sim}{u}(s) = u\underset{\sim}{e} + w\underset{\sim}{n}$ we define a linearized strain ϑ and a linearized rotation φ analogous to the shell theory

[8]:

$$\vartheta = \underset{\sim}{e}\cdot\underset{\sim}{u}' = u' - \sigma w \ , \quad \varphi = \underset{\sim}{n}\cdot\underset{\sim}{u}' = w' + \sigma u \ . \tag{8.1}$$

Introducing (8.1) into (7.1) yields the generalized strain measures in terms of the linearized quantities ϑ and φ. It follows that among equivalent strain measures the kinematical relations take the simplest form for $k = 1$, $l_1 = 3/2$ and $l_2 = 2$:

$$\begin{aligned} \gamma &= \vartheta + 1/2\vartheta^2 + 1/2\varphi^2 \ , \\ \varkappa &= -\varphi' - \varphi'\vartheta + \varphi\vartheta' \ , \end{aligned} \tag{8.2}$$

The corresponding static equations may be obtained from the principle of virtual work or by specialization of the general equations (4.23) and (5.13). Thus the equilibrium equations are

$$\begin{aligned} &[(1 + \vartheta)N - \varphi'M - (\varphi M)']' - \sigma[M' + \varphi N + \vartheta'M + (\vartheta M)'] + p_u = 0 \ , \\ &[M' + \varphi N + \vartheta'M + (\vartheta M)']' + \sigma[(1 + \vartheta)N - \varphi'M - (\varphi M)'] + p_w = 0 \ . \end{aligned} \tag{8.3}$$

Here N, M are the conjugate stresses and couple resultants related to the strains (8.2) by the constitutive equations (6.18) and p_u, p_w denote the components of the load vector $\underset{\sim}{p}$ with respect to the undeformed base $\{\underset{\sim}{e},\underset{\sim}{n}\}$. The corresponding boundary conditions have the form

$$\begin{aligned} &(1 + \vartheta)N - \varphi'M - (\varphi M)' = F^*_u \quad &&\text{or} \quad u = u^* \\ &M' + \varphi N + \vartheta'M + (\vartheta M)' = F^*_w \quad &&\text{or} \quad w = w^* \\ &(1 + 2\gamma)\sqrt{1 + (\varphi/1 + \vartheta)^2}\, M = H^* \quad &&\text{or} \quad n_\nu = n_\nu^* \end{aligned} \tag{8.4}$$

where $\underset{\sim}{F}^*(s_q) = F^*_u\underset{\sim}{e} + F^*_w\underset{\sim}{n}$, $q = 1,2$, and H^* is defined by (5.12). The rotational boundary parameter n_ν (5.14) may now be expressed in terms of the linearized quantities (8.1). It should be pointed out that within the assumption of the first approximation theory the derived variant of the rod theory is the simplest one for unrestricted rotations and/or displacements. In the particular case of straight beams an analogous theory had been derived by EPSTEIN and MURRAY [33].

For many purposes it is worthwhile to examine possible simplifications resulting from suitable restrictions imposed on the rod deformation. As in the case of general shells [8,12-17], simplified variants for rods may be formulated utilizing the estimation (6.12) for the strain energy function, which implies that terms in the extensional and flexural strain measures of $O(\eta\theta^2)$ and of $O(\eta\theta^2/h)$, respectively, may be omitted. To obtain a precise estimation of various terms in the strain-displacement relations we first note that the linearized quantities and their derivatives (8.1) can be expressed by (2.11) in terms of the nominal strain $\varepsilon = \lambda - 1$ and the angle of rotation ω

$$\vartheta = (1 + \varepsilon) \cos \omega - 1 \quad , \quad \varphi = (1 + \varepsilon) \sin \omega \quad , \tag{8.5}$$

$$\begin{aligned} \vartheta' &= -(1 + \varepsilon)\, \omega' \sin \omega + \varepsilon' \cos \omega \,, \\ \varphi' &= (1 + \varepsilon)\, \omega' \cos \omega + \varepsilon' \sin \omega \,. \end{aligned} \tag{8.6}$$

Furthermore the following estimations hold

$$\omega' = O(\eta/h) \quad , \quad \varepsilon' = O(\varepsilon/L) \,. \tag{8.7}$$

To obtain $(8.7)_1$ we note that the measure of change of curvature $\varkappa$ can be expressed by $(2.4)_3$ and (2.5) as $\varkappa = -(1 + \varepsilon)^2\omega'$. Thus $(6.13)_2$ and (6.14) imply $(8.7)_1$. The estimation $(8.7)_2$ follows from (6.14) and the definition of the wave length of deformation pattern [4].

In view of the assumption of small strains it follows from (8.5-6) that the strain-displacement relations (8.2) can be simplified further if the magnitude of rotation is additionally restricted. Using the relations (8.5-6) the non-linear contribution in $(8.2)_2$ can be expressed as

$$-\varphi'\vartheta + \varphi\vartheta' = (1 + \varepsilon)\, \omega'(\cos \omega - 1 - \varepsilon) + \varepsilon' \sin \omega \tag{8.8}$$

Thus, within the classification scheme proposed by PIETRASZKIEWICZ [12,14] the expression $(8.2)_2$ for the bending strain may be simplified to the linear form, if $\omega = O(\theta^2)$, i.e. for the case of moderate rotations. Under this restriction it follows from (8.5) and (6.14) that $\vartheta^2 = O(\eta\theta^2)$. Consequently, within the accuracy of the strain energy (6.12) the strain-displacement relations of the theory of rods undergoing moderate rotations take the reduced form

$$\gamma = \theta + 1/2\,\varphi^2 \quad , \; \varkappa = -\,\varphi' \quad . \tag{8.9}$$

with the corresponding rotational boundary parameter

$$n_\nu(\underset{\sim}{u}) = -\,\varphi + O(\eta\theta) \; . \tag{8.10}$$

The linear approximation for n_ν in (8.10) is consistent with the strain-displacement relations (8.9). Indeed, inserting (8.9) into the principle of virtual work and applying standard variational calculation one gets the following form of the equilibrium equations

$$\begin{aligned} &N' - \sigma(M' + \varphi N) + p_u = 0 \; , \\ &M'' + \sigma N + (\varphi N)' + p_w = 0 \; , \end{aligned} \tag{8.11}$$

and associated boundary conditions

$$\begin{aligned} N &= F_u \quad \text{or} \quad u = u^* \; , \\ M' + \varphi N &= F_w \quad \text{or} \quad w = w^* \; , \\ M &= H^* \quad \text{or} \quad \varphi = \varphi^* \; , \end{aligned} \tag{8.12}$$

This variant of the rod theory is the one-dimensional counterpart of the non-linear SANDER-KOITER theory of shells [8,11]. Its linearization leads to the one-dimensional counterpart of the "best" linear theory of shells [1].

Whenever loads acting on the rod are conservative the equilibrium equations and static boundary conditions for the theory of unrestricted as well as for moderate rotations are Euler-Lagrange equations of the variational problem (7.15). Comparing both theories two features are of considerable importance. In the case of unrestricted rotations the strain energy is the functional of the displacement components u, w and their derivatives up to second order, whereas for the case of moderate rotations the strain energy does not depend on the second derivative of the tangential displacement u. This implies corresponding continuity requirements on the shape functions, when the finite element method is applied. Furthermore for unrestricted rotations the third (rotational) geometric boundary condition is non-linear with respect to the displacements and it becomes linear in the case of moderate rotations.

A further simplification of the strain-displacement relations (8.9) may be achieved under the additional assumption on the wave length of deformation pattern, which leads to the classical theory of shallow rods presented in many papers.

It is worthwhile to note that variants of the nonlinear theories of rods undergoing unrestricted or moderate rotations and the simplified theory of shallow rods discussed above are the only consistent ones. Indeed, introducing the assumption of small strains into (7.1) the generalized strain measures in terms of the displacements take the form

$$\begin{aligned} \tilde{\gamma} &= \vartheta + (k - 1/2)\vartheta^2 + 1/2\varphi^2 + (k - 1)\vartheta(\vartheta^2 + \varphi^2) + \\ &\quad + (k - 1)(\vartheta^4 + 2\vartheta^2\varphi^2 + \varphi^2)/4 + O(\varepsilon^3) \\ \tilde{\varkappa} &= -\varphi' + \varphi\vartheta' - (l_1 - 1)\varphi'\vartheta - (2l_1 - l_2 - 1)\sigma(\vartheta + 1/2\vartheta^2 + 1/2\varphi^2 - \\ &\quad - (2l_1 - 3)[1/2\varphi'\varphi^2 + 3/2\varphi'\vartheta^2 - \varphi\vartheta\vartheta' + 1/2(\vartheta^2 + \varphi^2)(\varphi'\vartheta - \varphi\vartheta') + O(\eta\varepsilon^2/h,\ \varepsilon^2/R) \ . \end{aligned} \tag{8.13}$$

It follows that for any choice of the constants k, l_1, l_2 different from $k = 1$ $l_1 = 3/2$, $l_2 = 2$, i.e. for any equivalent strain measures different from (8.2) the kinematical relation take a much more complicated form. Furthermore the expressions (8.13) still contain terms, which can be ommited within the accuracy of the first approximation theory. In fact using (8.5-7) it can be shown that for any choice of the strain measures the strain-displacement relations can be simplified to the form (8.2).

9. COMPARATIVE ANALYSIS. CONCLUSIONS

In the previous chapter we pointed out that within the first approximation theory only three variants, describing the rod boundary value problem based on different levels of nonlinearity, are fully consistent. In the case of general shells the situation is less obvious. A wide variety of nonlinear shell theories had been proposed in the literature. In this chapter we shall show that partial verification of the consistency or inconsistency of shell theories can be achieved by reducing of the basic shell equations to their one-dimensional form describing the cylindrical bending of shells. Accordingly, various geometrically nonlinear shell theories known in the literature will be considered. For later reference the corresponding strain-displacement relations in their reduced one-dimensional form are listed in Tab. 3.

	strain-displacement relations	shell (rod) theory
S1	$\gamma = \theta + 1/2\theta^2 + 1/2\varphi^2$ $\varkappa = -\varphi' - \varphi'\theta + \varphi\theta'$	"large rotations", Nolte and Stumpf [16] "large rotations", Pietraszkiewicz [13] (theory of unrestricted rotations for rods)
S2	$\gamma = \theta + 1/2\varphi^2$ $\varkappa = -\varphi'$	"small finite deflections", Sanders [11], Koiter [8] (theory of moderate rotations for rods)
S3	$\gamma = \theta + 1/2\varphi^2$ $\varkappa = -\varphi' - \sigma\theta$	"moderate rotations", Pietraszkiewicz [12,14]
S4	$\gamma = \theta + 1/2\theta^2 + 1/2\varphi^2$ $\varkappa = -\varphi' - \sigma\theta - 1/2\sigma\varphi^2 - 1/2\varphi'\varphi^2$	"large rotations", Schmidt [15]
S5	$\gamma = \theta + 1/2\theta^2 + 1/2\varphi^2$ $\varkappa = -\varphi' - \sigma\theta - \varphi'\theta - 1/2\sigma\varphi^2 - \varphi'\varphi^2$	"large rotations", Pietraszkiewicz [14]
S6	$\gamma = \theta + 1/2\theta^2 + 1/2\varphi^2$ $\varkappa = -\varphi' - \sigma\theta + \varphi\theta' - 1/2\sigma\varphi^2$	"moderately large rotations", Basar [19], Harte [18]
S7	$\gamma = \theta + 1/2\varphi^2$ $\varkappa = -\varphi' - \sigma\theta + \varphi\theta' - 1/2\sigma\varphi^2$	"large deformation and rotations", Yaghmai [21]**)
S8	$\gamma = \theta + 1/2\theta^2 + 1/2\varphi^2$ $\varkappa = -\varphi' - \varphi'\theta - 1/2\sigma\varphi^2$	"large displacements", Varpasuo [22]**)
**) theory of axisymmetric shells of revolution		

Table 3. Shell strain - displacement relations in one-dimensional reduced form

In order to provide some informations about the accuracy of various theories let us consider the error in the associated strain measures defined by

$$e_\varepsilon = \varepsilon - \tilde{\gamma} \, , \quad e_\varkappa = h(\mu - \tilde{\varkappa}) \, , \qquad (9.1)$$

where we have chosen ε and μ as the (exact) reference strain measures and h, $\tilde{\gamma}$, $\tilde{\varkappa}$ denote here the rod thickness and any strain measures, the error of which we want to analyse. Using now the relations (8.7-8) and the estimation (8.9) the errors (9.1) can be obtained for any strain measures according to Tab. 3. Exemplarily let us present the results for the moderate rotation theory S2:

$$e_\varepsilon = 2\sin^4\omega/2 + 2\,\varepsilon\sin^2\omega/2\cos^2\omega/2\ , \quad e_\varkappa = 2\eta\sin^2\omega/2 + \varepsilon/hL\sin\omega . \qquad (9.2)$$

It is seen that the errors in the simplified strain expressions depend mainly on the magnitude of the rotation and in a less significant way on the extensional strain and on the ratio thickness over wavelength of deformation pattern. A complete graphic representation of relative errors for different forms of flexural strains is given in Fig. 4. Therefore and with the help of the aforementioned estimation of corresponding extensional strains the range of applicability for various theories may be outlined. In particular structural rotations of problems described by the moderate rotation theory S2 should not exceed $10^\circ \div 15^\circ$.

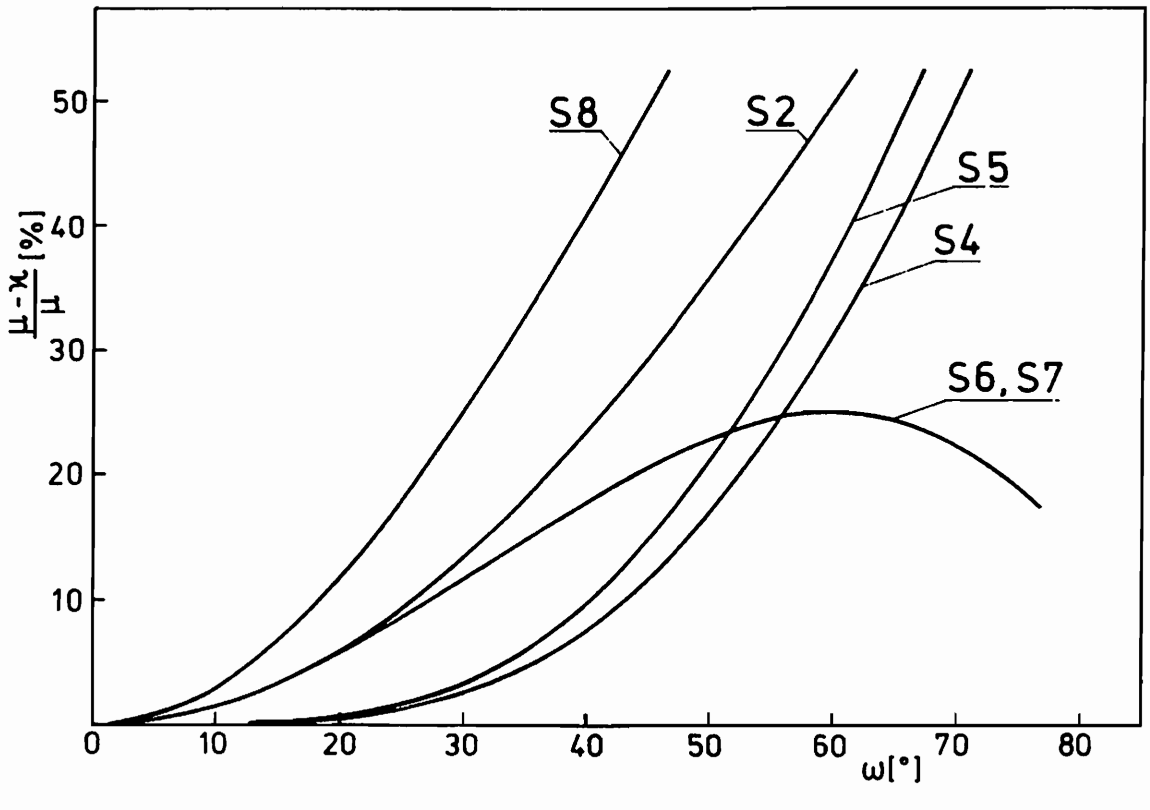

Fig. 4. Relative error for various shell strain - displacement relations

The given formulation of simplified theories is based on the assumption that small terms in the strain-displacement relations, i.e. with neglectible contributions to the energy expression, lead to small differences in the solution. However, this has been proved only in the linear case [1]. To give insight into this and related problems in the nonlinear case numerical calculations of various rod boundary value problems have been carried out.

Three structures loaded by a singular force, the classical elastica (Fig. 5), a moderately deep circular arch (Fig. 6a-d) and a deep circular arch (Fig. 7) have been analysed. The results were obtained by using a finite element method

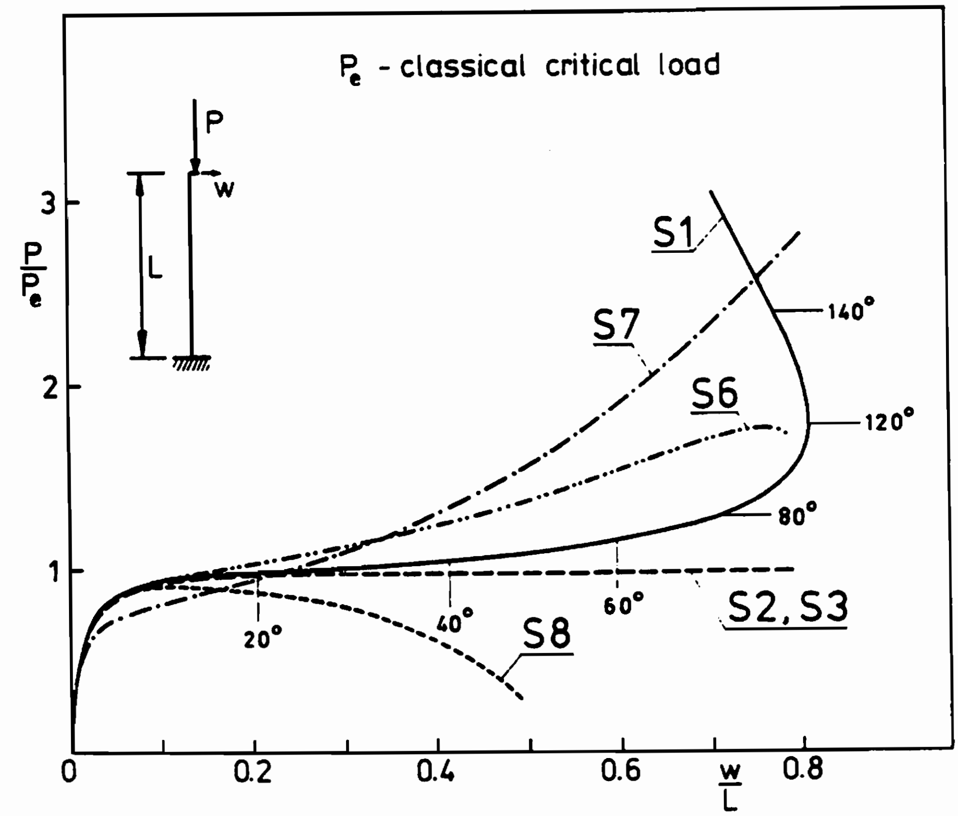

Fig. 5. Eulerian elastica: Load vs deflection predicated by different theories

with high-precision elements described in [17,42]. Since our ultimate aim is to compare solutions of various theories a comprehensive study of the accuracy of numerical solutions has been carried out and a comparsion with analytical and numerical solutions available in the literature has been made. In particular our results based on the simplest exact theory S1 are in full agreement with the analytical solutions of an extensible [43] and inextensible [46] theory for finite deflections of rods (Fig. 6a,7). For two examples, moderately-deep and deep arch (Fig.6a,7) the strain energy due to extensional deformation is less than 0.5% of the total strain energy. It is worthwhile to mention further that the solution obtained by applying the one-dimensional finite rotation theory S1 differs very slightly from a FE solution of the three-dimensional elasticity theory published in [44] (Fig. 6a,7).

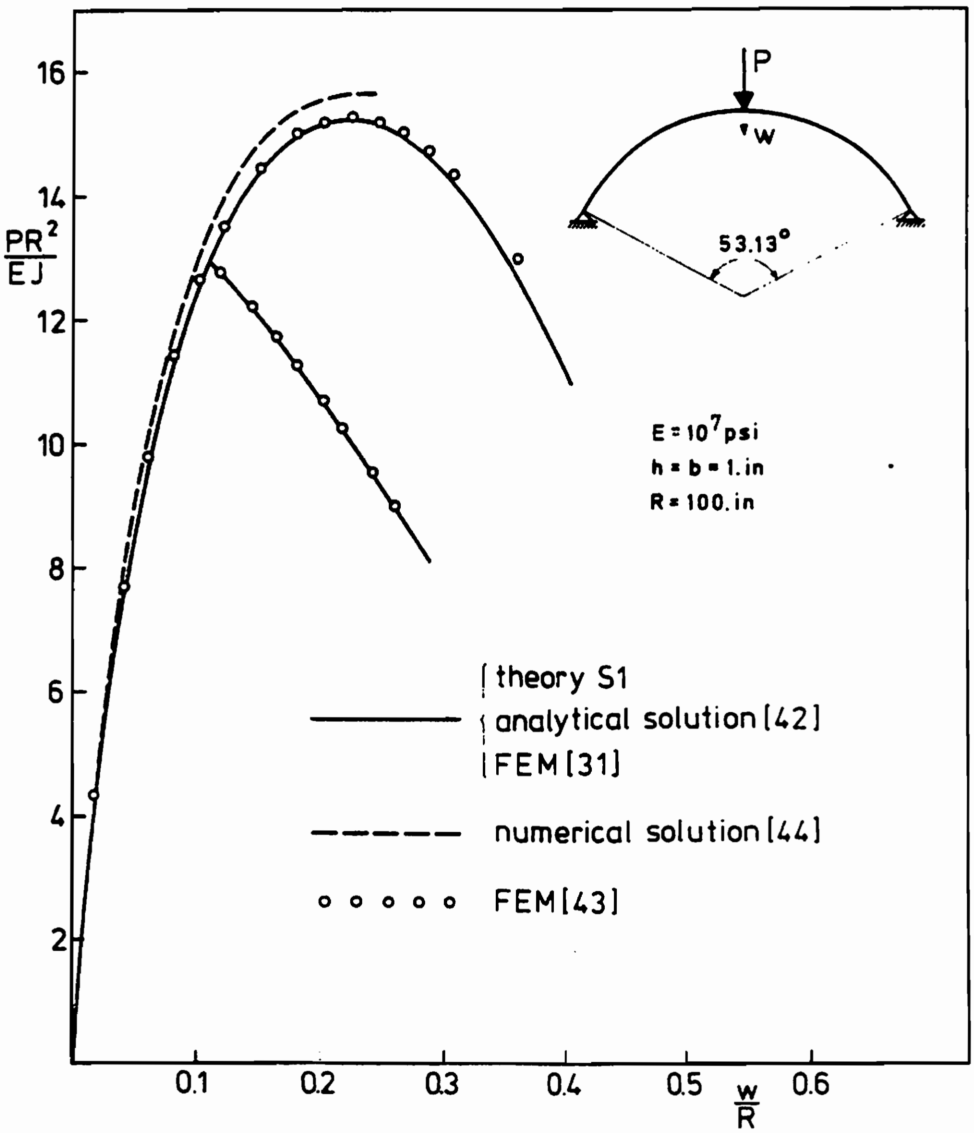

Fig. 6a. Moderately deep arch: Load vs deflection predicated by different finite deflection theories

All three analysed problems are highly geometrically non-linear and therefore it is obvious that the consistent theory of moderate rotations S2 may reflect the real structural response only in a limited range of loading. It should be pointed out that significant differences between the solutions of moderate and finite rotation theory arise at load levels with maximum rotations about 10 + 20^{o} (Fig. 5,6b,7). This observation corresponds to the results of theoretical error estimates carried out before. Furthermore it is remarkable that the difference in the solutions for moderate and finite rotation theory are not only of quantitative but also of qualitative nature. This may be observed in all examples but the case of the moderately-deep arch (Fig. 6b) is of particular interest. The

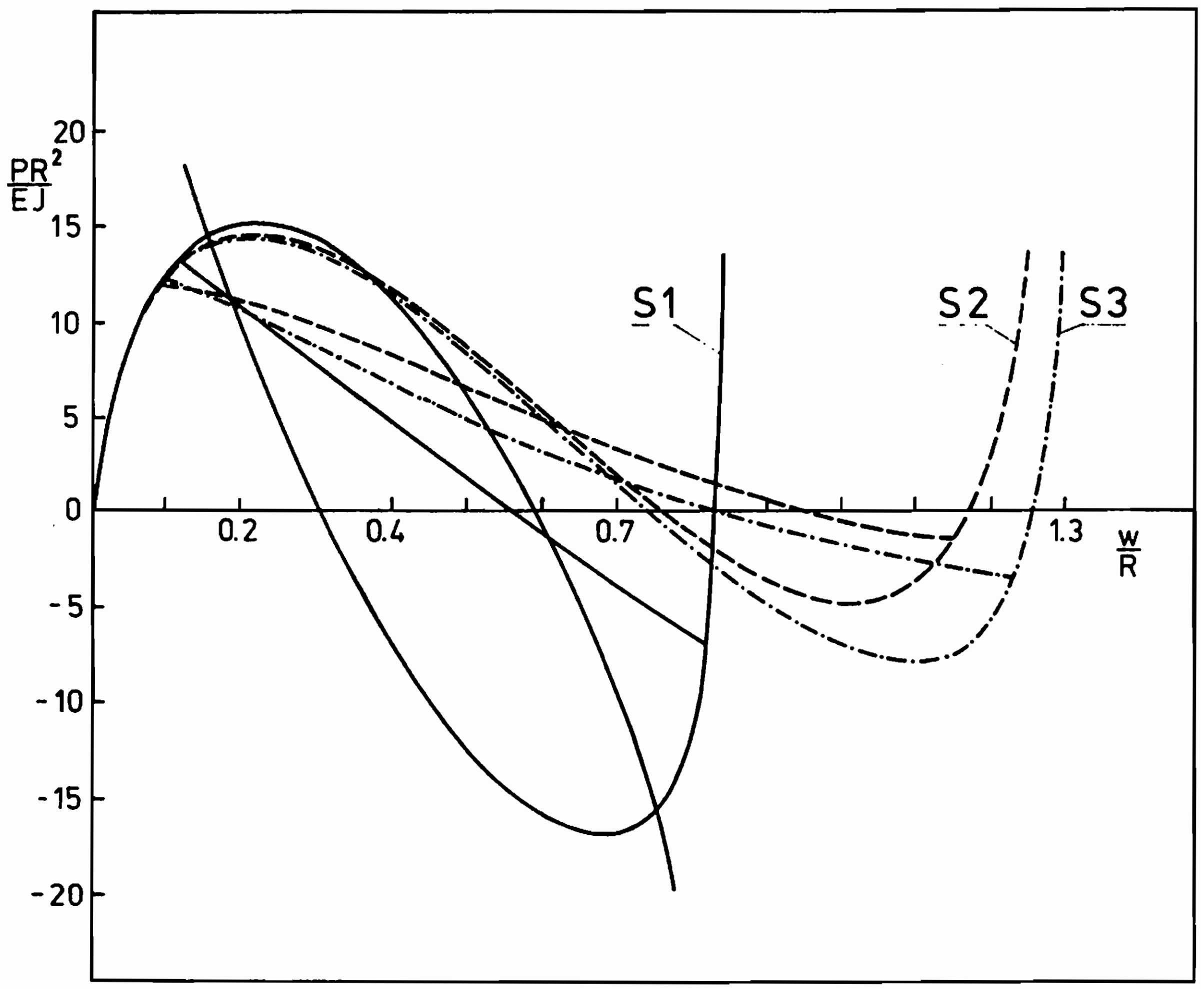

Fig. 6b. Moderately deep arch: Load vs deflection - Analysis of moderate rotation shell theories

real behaviour of this structure indicates the so-called "looping-effects" [17], i.e. the equilibrium path contains "loops" with many extremum points (in Fig. 6a-d only parts of the equilibrium path are shown), whereas the solution for the moderate rotation theory does not display these phenomena. During the numerical analysis of various examples the modified moderate variant S3 (Tab.3) leads within the range of validity to neglectible deviations in the solutions. Therefore it does not seem to be meaningful to retain the additional term $\sigma\vartheta$ in the flexural strain expression (in contrary to the statement made in [15]).

Within the range of higher nonlinearity refined shell theories are characterized by nonlinear flexural strain measures. Beside the large rotation shell theory given recently by the authors [16] with strain measures identical to the simplest general rod theory (8.2) six additional variants have been analysed (Tab. 3).

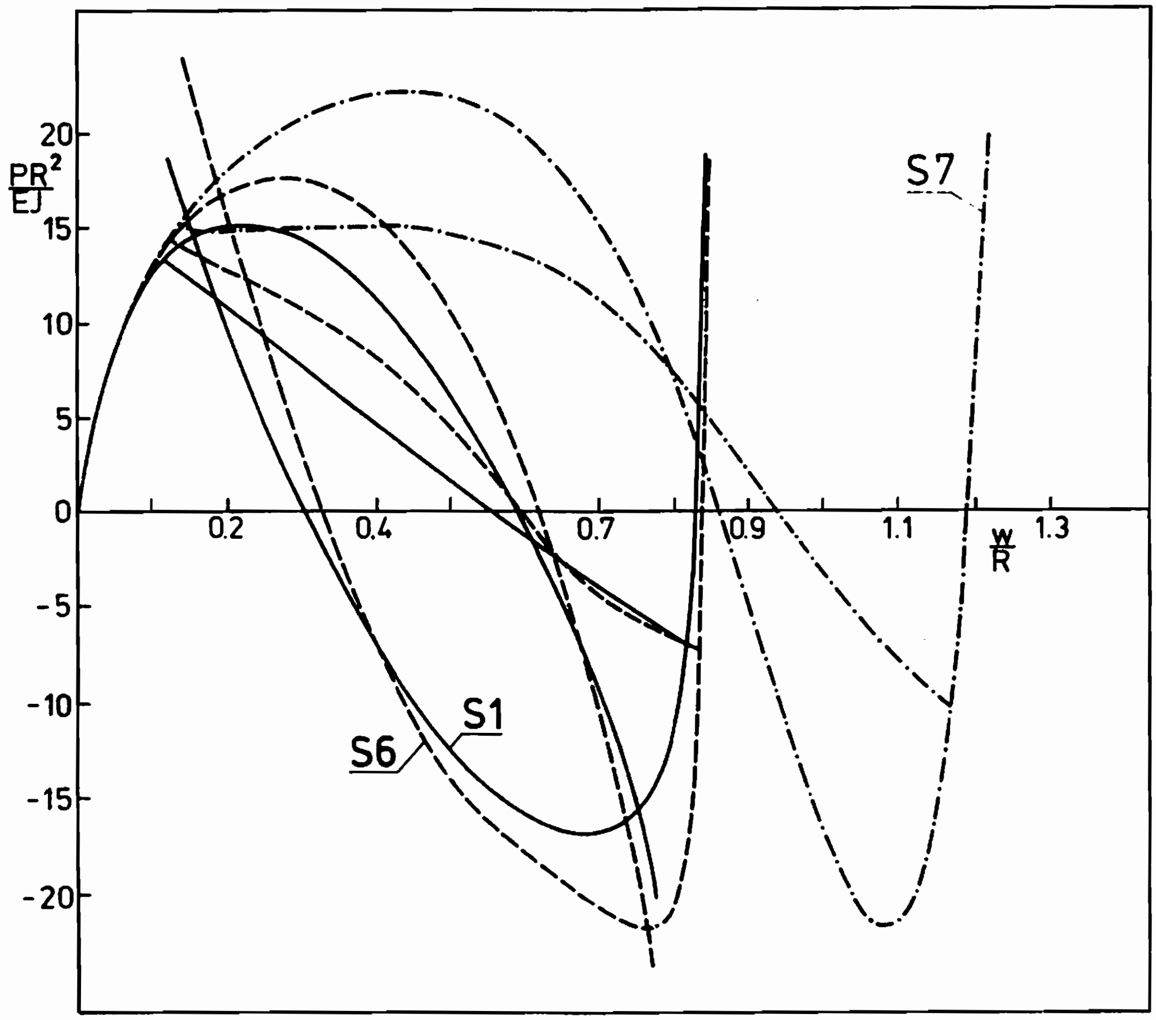

Fig. 6c. Moderately deep arch: Load vs deflection - Analysis of refined shell theories

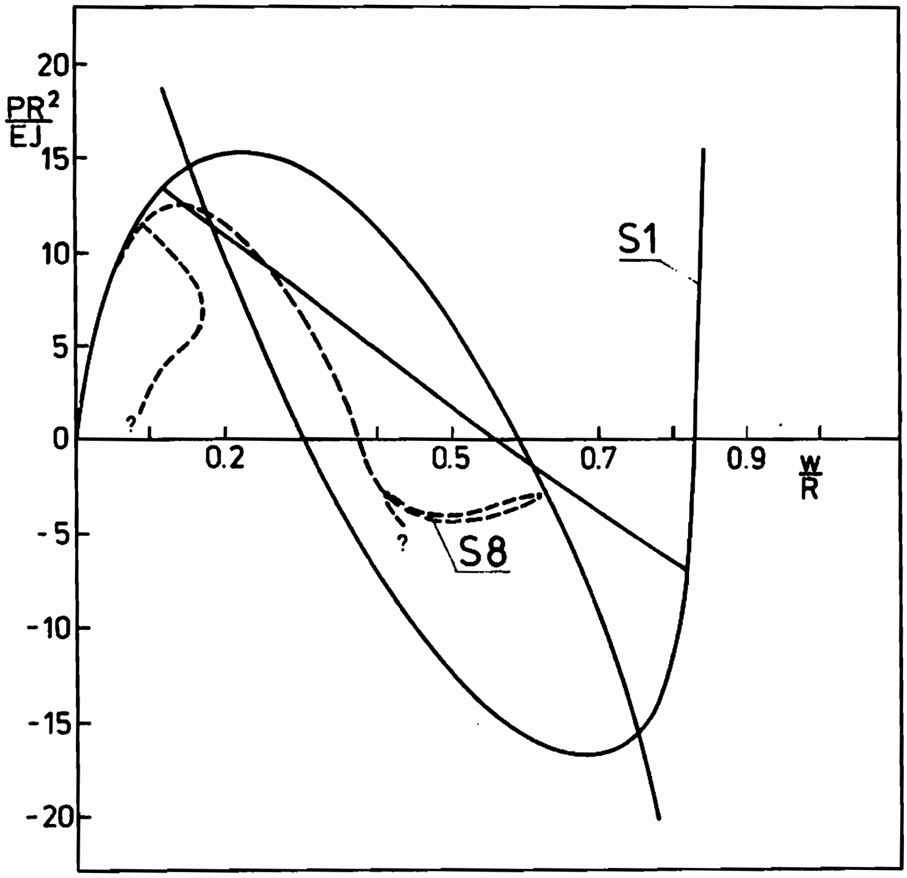

Fig. 6d. Moderately deep arch: Load vs deflection - Analysis of refined shell theories

Apart from various quantitative differences along the load-displacement path the theories S6 [18,19] and S7 [21] predicate critical loads with remarkable deviations to the reference solution S1 (Fig. 5, 6c). Additionally S7 yields unsatisfactory results in the postbuckling range according to Fig. 6c, due to an extensional measure identical with the moderate rotation theory S2. Therefore the theories S6 and S7, in spite of more retained terms in the strain measures, cannot be treated as consistent improvements of the moderate rotation theory. Moreover theory S8 shows the influence of inconsistent simplifications of the strain measures to the stability of the solution algorithms. Not only the results differ strongly from the exact ones, but also no convergence of the iterative procedure has been obtained at points denoted by a question sign in

Fig. 6d. It seems that the corresponding equilibrium equations are ill-conditioned.

Recently variants of large rotation shell theories have been formulated in [14,15] here denoted by S4 and S5 (Tab. 3). When compared with the results given in [13,17,23], formal transformations under the additional demand of a greater

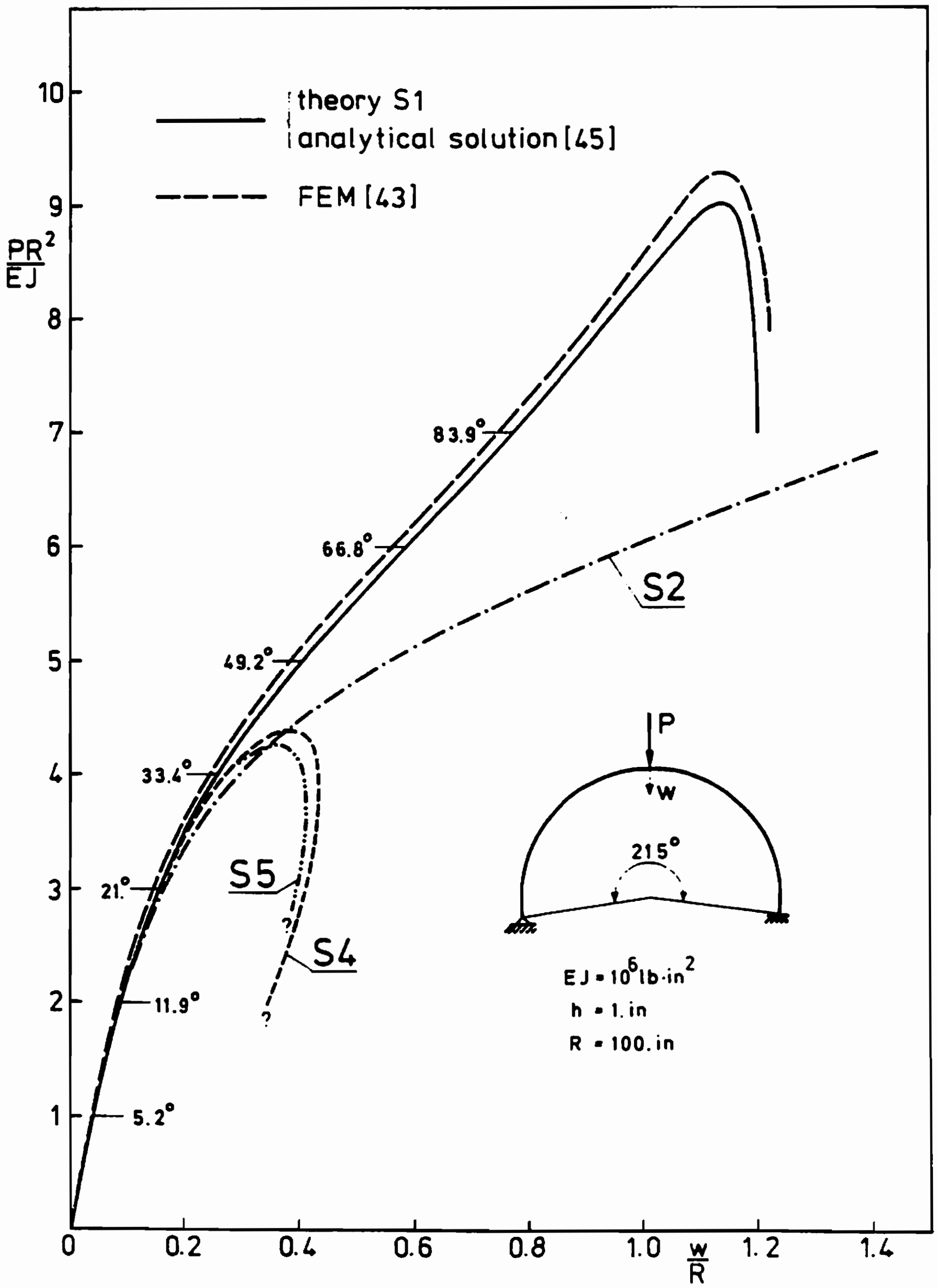

Fig. 7. Deep arch: Load vs deflection predicated by different theories

error margin in the strain energy expression lead to a non-dependency of the strain energy on second derivatives of tangential displacements. Subsequently convergence problems during the iteration process, as in the case S8, appear in the large rotation range. Apart from this by strong quantitative and qualitative deviations in the prediction of the nonlinear structural response (Fig. 7) these shell variants, though formally correct in the sence of [12], cannot be regarded as a consistent extension of the moderate rotation theory. Moreover numerical results based on refined shell variants S4 + S8 again harmonize with the error estimates represented in Fig. 4.

The theoretical investigations of previous chapters and the numerical results of this section leads to the following conclusions:

a) within the first approximation rod theory an appropriate choice of the strain measures yields a simple form of the basic equations valid for small strains and unresticted displacements and/or rotations;

b) formulations of simplified variants of rod theories require a rigorous approach; in particular an estimation term by term in the strain-displacement- and other relations can lead to inconsistances;

c) there are rod problems for which the solutions according to the unrestricted or moderate rotation rod theory and various variants containing nonlinear change of curvature expressions differ not only quantitatively but also qualitatively;

d) rod theories which are not consistent can lead to numerical instabilities when approximation procedures are applied.

Finally let us point out that equivalent problems arise also in the nonlinear theories of general shells and the concluding remarks of this paper are valid also in this case [23].

REFERENCES

1. W.T. Koiter and J.G. Simmonds, Foundations of shell theory. Proc. 13th Int. Congr. Theor. Appl. Mech., Moscow 1972, Springer-Verlag Berlin (1973).

2. F. John, Refined interior shell equations. Proc. 2nd IUTAM Symp.: Theory of Thin Shells. Copenhagen, September 5.-9., 1967. Springer-Verlag Berlin (1967).

3. W.T. Koiter, The intrinsic equations of shell theory with some applications. Mechanics Today (ed. S. Nemat-Nasser), vol. 5, Pergamon Press (1980).

4. W.T. Koiter, A consistent first approximation in the general theory of thin elastic shells. Proc. IUTAM Symp. Theory of Thin Shells. North-Holland P.Co., Amsterdam (1960).

5. J.G. Simmonds and D.A. Danielson, Nonlinear shell theory with a finite rotation vector. Proc. Kon. Ned. Ak. Wet. B73, 460 (1970).

6. S.N. Atluri, Alternate stress and conjugate strain measures, and mixed variational formulations involving rigid rotations for computational analysis of finitely deformed solids, with application to plates and shells - part I: theory. Report GIT - CACM - SNA - 81-34. Georgia Institute of Technology, Atlanta (1981).

7. E. Reissner, On axisymmetric deformations of thin shells of revolution. Proc. 3rd Symp. Appl. Math., New York (1950).

8. W.T. Koiter, On the nonlinear theory of thin elastic shells. Proc. Kon. Ned. Akad. Wet. B69, 1 (1966).

9. B. Budiansky, Notes on nonlinear shell theory. Trans. ASCE, J. Appl. Mech. 5, 393 (1968).

10. W. Pietraszkiewicz and M.L. Szwabowicz, Entirely Lagrangian nonlinear theory of thin shells. Arch. Mech. 33, 273 (1981).

11. J.L. Sanders, Nonlinear theories for thin shells. Quart. Appl. Math. 21, 21 (1963).

12. W. Pietraszkiewicz, Finite rotations in the nonlinear theory of thin shells.

In: Thin Shell Theory, New Trends and Applications (Ed. H. Olszak), Springer, Vien (1980).

13. W. Pietraszkiewicz, A simplest consistent version of the geometrically nonlinear theory of elastic shells undergoing large/small rotations. ZAMM 63, T 200 (1983).
14. W. Pietraszkiewicz, Lagrangean description an incremental formulation in the non-linear theory of thin shells, Int. J. Non-Linear Mech. 19, 115 (1984).
15. R. Schmidt, On geometrically non-linear theories for thin elastic shells. Proc. EUROMECH Colloquium: Flexible Shells, Theory and Applications. München, May 17.-20. (1983).
16. L.-P. Nolte and H. Stumpf, Energy-consistent large rotation shell theories in Lagrangean description. Mech. Research. Comm. 10, 213 (1983).
17. L.-P. Nolte, Beitrag zur Herleitung und vergleichende Untersuchung geometrisch nichtlinearer Schalentheorien unter Berücksichtigung grosser Rotationen. Ph.D. Dissertation, Ruhr-Universität Bochum, (1983).
18. R. Harte, Doppelt gekrümmte finite Dreieckelemente für die lineare und geometrisch nichtlineare Berechnung allgemeiner Flächentragwerke. Tech. wiss. Mitt. d. Inst. KIB, Nr. 82-10. Ruhr-Universität Bochum (1982).
19. Y. Basar, Eine geometrische nichtlineare Schalentheorie. Konstr. Ingenieurbau, Berichte Nr. 38/39, Ruhr-Univ. Bochum, (1981).
20. L.A. Shapavalov, Simple variant of equations of the geomtrically nonlinear theory of thin shells. Mech. of Solids 3, 56 (1968).
21. S. Yaghmai, Incremental analysis of large deformations in mechanics of solids with applications to axisymmetric shell of revolution. Ph.D. Dissertation, University of California. Berkeley (1969).
22. P. Vapasuo, Incremental analysis of axisymmetric shallow shell with varying strain-displacement relations. Comp. Meth. Appl. Mech. Engng.21, 153 (1980).
23. L.-P. Nolte, J. Makowski and H. Stumpf, On strain measures in the analysis of thin elastic shells undergoing large rotations, (in preparation.)

24. A.E.H. Love, A treatise on the mathematical theory of elasticity. 4th ed., Dover, New York (1944).

25. S.S. Antman, The theory of rods. Handbuch der Physik, vol. VIa 12, Springer-Verlag (1972).

26. I. Tadjabakhsh, The variational theory of the plane motion of the extensible elastica. Int. J. Engng. Sci. 4, 433 (1966).

27. S.S. Antman, General solutions for plane extensible elasticae having nonlinear stress-strain laws. Quart. Appl. Math. 26, 35 (1968).

28. E. Reissner, One-dimensional finite-strain beam theory: the plane problem. ZAMP 23, 795 (1972).

29. S.S. Antman, Qualitative theory of the ordinary differential equations of nonlinear elasticity. Mechanics Today (ed. S. Nemat-Nasser), vol. 1, Pergamon Press, New York (1972).

30. G. Wempner, Mechanics of solids with applications to thin bodies. Mc. Graw-Hill Book Co., New York (1973).

31. K. Brink and W.B. Krätzig, Geometrically correct formulations for curved finite bar elements under large deformations. Non-linear Finite Element Analysis in Structural Mechanics (Eds. W. Wunderlich, E. Stein, K.-I. Bathe), Springer-Verlag. New York (1981).

32. E. Stein, Incremental methods in finite elasticity, especially for rods, Proc. IUTAM Symp. "Finite Elasticity" Martinus Nijhoff Publishers, The Hague, Boston, London, 379 (1982)

33. M. Epstein and D.W. Murray, Large deformation in-plane analysis of elastic beams. Comp. Struct. 6, 1 (1976).

34. M. Pignataro, A. Di Carlo and R. Casciaro, On nonlinear beam models from the point of view of computational post-buckling analysis. Int. J. Solids Structures 18, 327 (1982),

35. A. Libai and J.G. Simmonds. Large-strain constitutive laws for the cylindrical deformation of shells. Int. J. Non-Linear Mech. 16, 91 (1981).

36. R. Hill, Aspects of invariance in solid mechanics. Advances in Applied Mechanics (ed. Chia-Shun Yih), vol. 18. Academic Press, New York (1978).

37. J.G. Simmonds. The strain energy density of rubber-like shells. DAMACS Report 83-07, University of Virginia (1983).

38. I.-P. Zürcher and W. Schumann, Some intrinsic considerations on the non-linear theory of thin shells. Acta Mech. 40, 123 (1981).

39. R.W. Ogden, Elastic deformation of rubberlike solids. Mechanics of Solids (R. Hill 60th Anniversary Volume), Pergamon Press. Oxford (1982).

40. M.J. Sewell, On configuration-dependent loading. Arch. Rat. Mech. Anal. 23, 327 (1967).

41. H. Stumpf, On the non-linear buckling and post-buckling analysis of thin elastic shells. Int. J. Non-Linear Mech. 19, 195 (1984).

42. L.-P. Nolte, R. Diekkämper and J. Makowski, Two efficient finite element formulations in the analysis of highly nonlinear deformation of thin elastic beams. ZAMM, Band 65, Heft 4/5, (1985).

43. J.V. Huddleston, Finite deflection and snap-through of high circular arches. J. Appl. Mech. 5, 763 (1968).

44. R.D. Wood and O.C. Zienkiewicz, Geometrically nonlinear finite element analysis of beams, frames, arches and axisymmetric shells. Comp. Struct. 7, 725 (1977).

45. P. Sharifi and E.P. Popov. Nonlinear buckling analysis of sandwich arches. J. Engng. Mech. Div., 1397 (1971).

46. D.A. Da Deppo and R. Schmidt, Instability of clamped-hinged circular arches subjected to a point load. J. Appl. Mech., 894 (1975).

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