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Geometrically non-linear
theories of thin elastic shells

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GEOMETRICALLY NON-LINEAR THEORIES

OF THIN ELASTIC SHELLS

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Summary

Basic relations of the non-linear theory of thin elastic shells are reviewed. Various sets of equilibrium equations and compatibility conditions as well as corresponding work-conjugate static and geometric boundary and corner conditions are derived in terms of displacements of the shell middle surface, in terms of rotations and other fields as well as in terms of two-dimensional strain and/or stress measures as independent variables. The final non-linear shell relations are consistently simplified under assumption of small strains. Displacement shell relations are simplified further under the consistently restricted rotations, while the intrinsic shell equations are consistently simplified for various ratios of extensional and bending strains. In the case of conservative surface and boundary loadings appropriate variational functionals are constructed for displacement and rotational shell equations. Apart from the comprehensive review of achievements in the field of derivation of various versions of the first-approximation geometricalaly non-linear theory of thin elastic shells several original contributions to the field are given as well.

Zusammenfassung

Eine Übersicht über die grundlgenden Beziehungen der nichtlinearen Theorie dünner elastischer Schalen wird zusammengestellt. Verschiedene Formen der Gleichgewichts- und Verträglichkeitsbedingungen sowie deren conjugierte statische und geometrische Rand- und Eckbedingungen werden hergeleitet, wobei entweder Verschiebungen der Schalenmittelfläche oder Rotationen zusammen mit anderen Feldgrößen oder zweidimensionale Dehnungs- und/oder Spannungsgrößen als unabhängige Variablen auftreten. Die resultierenden nichtlinearen Beziehungen werden unter der Voraussetzung kleiner Dehnungen konsistent vereinfacht. Die Verschiebungsbeziehungen der Schalentheorie werden durch konsistent begrenzte Rotationen weiter vereinfacht. Die Schalengleichungen, in denen die Dehnungs- und/oder die Spannungsgrößen als unabhängige Variablen auftreten, werden für verschiedenen Verhältnisse der Membrandehnungen zu den Biegedehnungen ebenfalls vereinfacht. Für konservative Flächen- und Randbelastungen werden geeignete Variationsfunktionale in Verschiebungen oder Rotationen konstruiert. Neben einer umfassenden Übersicht über die Fortschritte auf dem Gebiet der Herleitung verschiedener Versionen der ersten Approximation der geometrisch nichtlinearen Theorie dünner elastischer Schalen werden auch mehrere neue Beiträge vorgestellt.

Р е з ю м е

Дано обсуждение основных зависимостей нелинейной теории тонких упругих оболочек. Рассмотрены различные виды уравнений равновесия и условий совместности деформации а также соответственные энергетически согласованные статические и геометрические краевые условия и условия в угловых точках края оболочки. Эти основные системы зависимостей выражены через перемещения срединной поверхности оболочки или через повороты и другие параметры или же через двумерные меры деформаций и/или напряжений как независимые переменные. Разрешающие системы нелинейных уравнений теории оболочек соответственно упрощены при предположении что деформации всюду малы. Уравнения оболочек в перемещениях дополнительно упрощены при ограничении величины поворотов, а уравнения оболочек в мерах деформаций и/или напряжений дополнительно упрощены при предположении различных соотношений мембранной и изгибной деформации. В случае консервативной поверхностной и краевой нагрузки построены соответственные вариационные функционалы для теории оболочек в перемещениях или в поворотах и других параметрах. Кроме обстоятельного обзора достижений в области построения различных вариантов нелинейной теории первого приближения тонких упругих оболочек, в работе представлен также ряд новых результатов автора в этой области.

S t r e s z c z e n i e

W pracy przedstawiono podstawowe zależności nieliniowej teorii powłok sprężystych. Rozważono różne postacie równań równowagi i warunków ciągłości odkształceń oraz odpowiednie energetycznie spójne statyczne i geometryczne warunki na brzegu i w narożach powłoki. Zależności te wyprowadzono przyjmując jako zmienne niezależne albo przemieszczenia powierzchni środkowej powłoki lub obroty i inne parametry lub też dwuwymiarowe miary odkształceń i/lub naprężeń. Końcowe nieliniowe zależności teorii powłok konsekwentnie uproszczono przy założeniu małych odkształceń. Przemieszczeniowe równania powłok są dodatkowo upraszczane przy konsekwentnie ograniczonych obrotach, podczas gdy równania powłok wyrażone przez miary odkształceń i/lub naprężeń są dodatkowo upraszczane przy założeniu różnych stosunków odkształceń giętnych i błonowych. W przypadku zachowawczych obciążeń powierzchniowych i brzegowych skonstruowano odpowiednie funkcjonały wariacyjne nieliniowej teorii powłok wyrażonej w przemieszczeniach lub w obrotach i innych parametrach. Oprócz wyczerpującego przeglądu w dziedzinie budowania różnych wariantów geometrycznie nieliniowej teorii pierwszego przybliżenia cienkich powłok sprężystych, praca zawiera również szereg nowych rezultatów autora w tej dziedzinie.

CONTENTS

1. Introduction

Shell theory attempts to describe the mechanical behaviour of a thin three-dimensional solid layer - the shell - by a finite number of fields defined over its reference (usually middle) surface. Since this is not possible, in general, the shell theory is an approximate one virtually by definition. It can not provide a complete and exact information about all three-dimensional fields describing the mechanical behaviour of the shell. However, the results which follow from such two-dimensional approximate description of the shell are usually sufficiently accurate for the majority of applications in science and technology. At the same time, the two-dimensional problem resulting from an appropriate shell theory is much easier to handle than the original three-dimensional one.

This report deals with one of the simplest formulations of the shell theory: the geometrically non-linear first-approximation theory of thin elastic shells. This theory is applicable when:

a) the shell is made of a homogeneous, isotropic and elastic material;

b) the shell is thin, i.e. $h/R \ll 1$, where h is the constant thickness of the undeformed shell and R is the smallest radius of curvature of its reference surface $\mathcal{M}$;

c) the undeformed reference surface is smooth, i.e. $(h/l)^2 \ll 1$, where l is the smallest wave length of geometric patterns of $\mathcal{M}$;

d) the shell deformation is smooth, i.e. $(h/L)^2 \ll 1$, where L is the smallest wave length of deformation patterns on $\mathcal{M}$;

e) the strains are small everywhere, i.e. $\eta \ll 1$, where η is the largest strain in the shell space.

Under an additional restriction of rotations of material fibres to be also small everywhere, the geomatrically non-linear theory reduces to the classical linear first-approximation theory of shells, which was discussed in detail in many papers and books, for example [135,296,79,83,175,275,75,39,158,85,228,26].

Within the assumptions given above the behaviour of an interior domain of the shell can be described with a sufficient accuracy by the behaviour of the shell reference surface. Already ARON [13] approximated the shell strain energy density by a sum of two quadratic functions describing the stretching and the bending of the shell reference surface. LOVE [135] came to the same conclusion by the application of two well known constraints, analogous to those used by KIRCHHOFF [111,112] in plate theory (cf. NOVOZHILOV [175]). The accuracy of such so called Kirchhoff-Love shell theory was examined in a number of papers [177,79,80,113,118,50,123,212,26]. In particular, NOVOZHILOV and FINKELSHTEIN [177] and KOITER [113] pointed out explicitly that within the basic assumptions the quadratic expression of LOVE [135] for the shell strain energy is a consistent first approximation. Moreover [113], various versions of the shell theory, which differ from the version given by LOVE [135] only by terms of the order of η/R in the definition of the two-dimensional measure of change of curvature, should be regarded as equivalent from the point of view of the first approximation to the shell strain energy. Since the consistently approximated strain energy of the shell is expressed entirely in terms of two-dimensional strain measures of the reference surface, the conclusions are valid both for the linear and for the geometrically non-linear theory of shells.

Although some geometric results about the non-linear deformation of the shell space had been given already by LOVE [135,Ch.24], DONNELL [54,55] and MUSHTARI [146,147] seem to be the first who proposed the simplest non-linear theory for stability analysis of cylindrical shells. MARGUERRE [144], MUSHTARI [148,149] and VLASOV [294] developed the non-linear theory of shallow shells which was applied with a great success to a number of problems of flexible shells analysed for strength, deformability and the loss of stability. In particular, applying Marguerre's theory KÁRMÁN and TSIEN [107] discovered that the axial compressive forces applied to a cylindrical shell drop considerably in the post-buckling range of deformation. This differed qualitatively from the behaviour of compressed bars and plates, but was in a good agreement with the experimental results for cylindrical shells. Many results obtained with the help of the non-linear theory of shallow shells have been summarized in the books of VLASOV

[296], VOLMIR [297,298], MUSHTARI and GALIMOV [157], KORNISHIN [121], BRUSH and ALMROTH [35], KANTOR [105] where further references may be found.

The foundations for the general geometrically non-linear theory of elastic shells were laid down by CHIEN [44]. He expanded all three-dimensional fields into series of the normal coordinate and applied order-of-magnitude estimates valid under the assumption of small strains. As a result, three equilibrium equations and three compatibility conditions were derived in [44] in an invariant tensor notation, which were then expressed in the intrinsic form, in terms of two-dimensional strains and changes of curvatures of the shell reference surface. Under additional assumptions about the thinness of the shell and the smallness of its curvature, 27 types of approximate versions of the intrinsic shell equations were given. It was assumed in [44] that when $h \to 0$ the limits of some functions do not change their order upon the surface differentiation. This assumption was criticized in [77,85] as to be applicable only to a limited class of shell problems. It was also recognized that only special problems can be formulated and solved directly in the intrinsic form. As a result, the very general approach of [44] has gained little attention in the following papers.

Alternative two-dimensional formulations of the non-linear theory of shells were given in an invariant tensor notation in the series of papers by MUSHTARI [150-154], GALIMOV [62-70] and ALUMÄE [4-8]. It was assumed there from the outset that the behaviour of the shell can be described with a sufficient accuracy by the behaviour of its middle surface. While Mushtari and Galimov presented several forms of shell relations in the natural bases of the undeformed and of the deformed surface, Alumäe derived his non-linear shell relations in the intermediate non-holonomic basis, which was obtained from the undeformed basis by its rigid-body rotation. Unfortunately, some of these original results were published in the local journals which even today are hardly available outside the Soviet Union. The monograph by MUSHTARI and GALIMOV [157] was written in the classical notation, using initially orthogonal system of coordinates coinciding with lines of principal curvatures of the undeformed surface. It provided well documented sets of the shell relations for simplified non-linear theory of medium bending and for the one of shallow shells. However, not all of the general results published in the original works of the authors were presented in their monograph with a sufficient generality and accuracy. In the classical notation some intermediate formulae became extremely complex and had to be simplified by omitting some terms which were supposed to be small. This raised some doubts

about the consistency and the range of applicability of the final relations of the geometrically non-linear theory of shell, cf. [115].

Various equivalent forms of the non-linear relations for thin shells were independently rederived and developed further by RÜDIGER [211], LEONARD [128], SANDERS [215], NAGHDI and NORDGREN [162], KOITER [115,116], WOŹNIAK [299], BUDIANSKY [36], SIMMONDS and DANIELSON [247,248], REISSNER [208] and PIETRASZKIEWICZ [182-185]. In particular, concrete error estimates given by JOHN [101,103] and BERGER [30] for the two-dimensional differential equations of the geometrically non-linear theory of elastic shells strengthened the foundations of the theory and established more precise bounds of its applicability. DANIELSON [49] and KOITER and SIMMONDS [120] worked out the refined intrinsic shell equations which were expressed in terms of internal stress resultants and changes of curvatures as independent field variables (cf. also [185,190]). SIMMONDS and DANIELSON [247,248] proposed the set of non-linear shell equations in terms of finite rotation and stress function vectors as independent variables and constructed an appropriate variational principle. PIETRASZKIEWICZ and SZWABOWICZ [201] derived entirely Langrangian non-linear shell equations in terms of displacements as independent variables. In case of dead surface and boundary loadings these equations were derivable as stationary conditions of the Hu-Washizu functional (cf. also [197]). Theory of finite rotations in shells developed by PIETRASZKIEWICZ [185,190] allowed then to work out the consistent classification of approximate versions of displacement equations for shells undergoing restricted rotations [195,197].

Various general theoretical aspects of the non-linear theory of thin shells are discussed also in the books by KILCHEVSKII [110], TEREGULOV [273], NAGHDI [159], GALIMOV [71], PIETRASZKIEWICZ [185,190], GRIGOLYUK and KABANOV [89], MASON [145], WOŹNIAK [300], WEMPNER [291], DIKMEN [53], ZUBOV [304], BERDICHEVSKIY [28], BAŞAR and KRÄTZIG [26], GALIMOV and PAIMUSHIN [73], CHERNYKH [42] and AXELRAD [15,17] as well as in the reviews or extensive papers by GOLDENVEIZER [78], KOITER [117], MUSHTARI [155,156], NOVOZHILOV [176], BAŞAR [20], LANGHAAR [127], PIETRASZKIEWICZ [187,191,193], KOITER and SIMMONDS [120], WOŹNIAK [301], SIMMONDS [243], NAGHDI [160,161], SCHMIDT and PIETRASZKIEWICZ [224], ATLURI [14], LIBAI and SIMMONDS [133], SCHMIDT [222], STUMPF [263] and SZWABOWICZ [271], where further references may be found. One-dimensional problems of the non-linear theory of elastic shells are extensively treated by SHILKRUT [237], SHILKRUT and VYRLAN [238], VALISHVILI [285], ANTMAN [10] and, in particular, by LIBAI and

SIMMONDS [134].

The behaviour of the shell near its lateral boundaries, i.e. in an edge zone of depth of the order of the shell thickness, is nearly always essentially three-dimensional. The physical explanation of this statement is quite simple. The external (or reactive) stresses applied to the shell lateral boundary surface are statically equivalent to the external force and moment resultants on the reference boundary contour plus some self-equilibrated part of the stress distribution over the lateral boundary surface. The resultants enter into the boundary conditions of the basic boundary-value problem which describes correctly the shell behaviour in its interior domain, far from its lateral boundary surfaces. The self-equilibrated part generates additional stresses in the shell space which are localized in the edge zone. Within the linear shell theory these additional stresses may be calculated approximately as some linear combinations of solutions of the plane and anti-plane problems for a semi-infinite strip [82,83] and then may be added to the basic stress state associated with the resultants, (cf. also [91]). An extension of this approximate method, based on a superposition of elementary stress states, to the non-linear range of deformation may not always be correct, in particular near the stress states associated with the bifurcation or limit points of solutions of the basic boundary value problem. Additionally, the exact stress distribution over the shell lateral boundary surface is rarely known in the majority of engineering problems, except in the case of a free edge. As a result, within the geometrically non-linear theory of shells little has been achieved in better two-dimensional description of the shell behaviour in the edge zone. Some approximate results have been given by KOITER and SIMMONDS [120] and NOVOTNY [172].

In this report basic relations of the non-linear theory of thin elastic shells are reviewed. Various consistent sets of the non-linear shell equations in terms of displacements, in terms of rotations and some other field variables as well as in terms of two-dimensional strain and/or stress measures as independent variables are discussed. The final non-linear relations are then consistently simplified under the assumption that strains are small, while displacement equations are simplified further under consistently restricted rotations. For some types of conservative surface and boundary loadings appropriate variational functionals are constructed for displacement and rotational non-linear shell equations.

During preparation of this report it became necessary to clearize some

theoretical problems which have not been fully treated in the literature. Among those new results is a discussion of integrability of kinematic boundary conditions, the construction of the general form of the work-conjugate static and geometric boundary conditions for displacement shell equations, alternative derivation of the rotation shell equations in the rotated and in the undeformed basis, the construction of the variational functional in terms of rotations, displacements and Lagrangian multipliers as well as an alternative derivation of the refined intrinsic shell equations.

The literature on various aspects of the non-linear theory of shells is very extensive and some kind of selection of references has to be made. The references in this report are given primarily to those original papers and monographs which deal with general aspects of the theory and are written in the invariant tensor notation. Other original papers and monographs, which are written in classical notation or which are dealing with special shell geometries, are referred on the basis of their historical or informative value. Although it is believed that the most important papers, which deal with the derivation of various invariant forms of non-linear shell relations, are included into the list of references, no attempt is made to provide the complete list of such references.

It is worthwhile to point out here once again that some of the two-dimensional relations of the non-linear theory of thin shells are derived by taking a difference between two groups of terms of the same order associated whith the deformed and undeformed reference surface. In the derivation process it often happens that the principal terms of those groups cancell out and the seemingly secondary terms are the only ones which appear in the final shell relations. In the geometrically non-linear theory of shells, in which strains are assumed to be always small, it is quite dangerous to simplify the intermediate relations by droping terms of the order of strains relative to the unity, since then the final relations may happen to be inconsistent or even incorrect. This has actually happened in several early papers devoted to the derivation and simplification of the geometrically non-linear theory of shells. In this report all two-dimensional relations associated whith the reference surface are derived for unrestricted strains. The small strain assumption is then used at the end of derivation process to simplify the final set of non-linear shell relations.

Stability analysis of flexible shells is one of the most important possible applications of the geometrically non-linear theory of shells discussed in this report. The literature on various approximate versions of the stability equations

for thin shells is extensive and has to be reviewed separately. The stability equations are usually derived as a result of superposition of two or more non-linear deformations of the shell. Since different types of approximation may be used to describe the first (basic) deformation and the following (superposed) deformations, a large variety of types of the shell stability equations for thin elastic shells may be constructed. We only note here that problems of super-postion of deformations and derivation of various types of stability equations have been discussed, among others, by NOVOZHILOV [173], KOITER [114], MUSHTARI [153], ALUMÄE [5,6], MUSHTARI and GALIMOV [157], TIMOSHENKO and GERE [278], DAREVSKII [52], VOLMIR [298], BOLOTIN [32,33], KOITER [116], BUDIANSKY [36], DANIELSON and SIMMONDS [51], SEIDE [229], ABÉ [1], BAŞAR [21], BRUSH and ALMROTH [35], ZUBOV [303], GRIGOLYUK and KABANOV [89], TALASLIDIS [272], van der HEIJDEN [92], STUMPF [257-263], SRUBSHCHIK [358], BAŞAR and KRÄTZIG [22,26], KRÄTZIG et al. [124,125], STEIN et al. [252,253], ECKSTEIN [59], ARBOCZ [11], NOLTE [164,165], SCHMIDT and STUMPF [225] and PIETRASZKIEWICZ [197], where further references are given.

It is not the aim of this report to review recent achievements in the large-strain non-linear theory of thin shells. Suffice it to point out that many two-dimensional relations collected here are applicable also to this more general case of shell deformation, provided that the behaviour of the shell is still approximated only by the behaviour of its reference surface. Such simple versions of the large-strain K-L type theories of shells were proposed recently by CHERNYKH [40,42], SIMMONDS [246] as well as by STUMPF and MAKOWSKI [264]. However, the change of the shell thickness during deformation should then be taken explicitly into account not only in the approximate form of the strain energy density but also in definitions of the external force and couple resultants applied to the shell reference surface and on its boundary contour. Outside of this review are also the more advanced models of shells in which the behaviour of the shell is described not only by the behaviour of its reference surface but also by additional higher-order independent parameters. We share the view expressed by KOITER and SIMMONDS [120] that the rapid development of numerical techniques in three-dimensional problems, in particular the finite element technique, may obviate the need of (complicated) refined shell theories in the near future.

2. Notation and geometric relations

The notation which will be used in this report follows that of KOITER [115] and PIETRASZKIEWICZ [185,190,193,197]. In order to make the paper self-contained we review here the notation and some basic geometric relations of the surface and its non-linear deformation.

Let $\mathcal{P}$ be the region of the three-dimensional Euclidean space $\mathcal{E}$ occupied by the shell in its undeformed configuration. In $\mathcal{P}$ we introduce the normal system of curvilinear coordinates $(\theta^1, \theta^2, \zeta)$ such that $-h/2 \leq \zeta \leq h/2$ is the distance from the middle surface $\mathcal{M}$ of $\mathcal{P}$ and h is the undeformed shell thickness, assumed here to be constant and small as compared to the smallest radius of curvature R of $\mathcal{M}$ and to the linear dimensions of $\mathcal{P}$.

The surface $\mathcal{M}$ is described by the position vector $\mathbf{r} = x^k(\theta^\alpha)\mathbf{i}_k$, $k=1,2,3$, $\alpha=1,2$, where $\mathbf{i}_k$ is an orthonormal basis attached to a point $O \in \mathcal{E}$. With each point $M \in \mathcal{M}$ we associate the natural covariant base vectors $\mathbf{a}_\alpha = \partial\mathbf{r}/\partial\theta^\alpha \equiv \mathbf{r}_{,\alpha}$, the covariant components $a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$ of the surface metric tensor $\mathbf{a}$ with the determinant $a = |a_{\alpha\beta}|$, the contravariant components $\varepsilon^{\alpha\beta}$ of the permutation tensor such that $\varepsilon^{12} = -\varepsilon^{21} = 1/\sqrt{a}$, $\varepsilon^{11} = \varepsilon^{22} = 0$, the unit normal vector $\mathbf{n} = \frac{1}{2}\varepsilon^{\alpha\beta}\mathbf{a}_\alpha \times \mathbf{a}_\beta$ and the covariant components $b_{\alpha\beta} = -\mathbf{a}_\alpha \cdot \mathbf{n}_{,\beta}$ of the curvature tensor $\mathbf{b}$. Contravariant components $a^{\alpha\beta}$ of $\mathbf{a}$, satisfying the relations $a^{\alpha\gamma}a_{\beta\gamma} = \delta^\alpha_\beta$, where $\delta^1_1 = \delta^2_2 = 1$, $\delta^1_2 = \delta^2_1 = 0$, are used to raise indices of the surface vectors and tensors, for example $a^\alpha = a^{\alpha\beta}a_\beta$, $b^\alpha_\beta = a^{\alpha\gamma}b_{\gamma\beta}$, etc.

The boundary contour $\mathcal{C}$ of $\mathcal{M}$ consists of the finite set of piecewise smooth curves given by $\mathbf{r}(s) = \mathbf{r}[\theta^\alpha(s)]$, where s is the arc length along $\mathcal{C}$. With each regular point $M \in \mathcal{C}$ we associate the unit tangent vector $\mathbf{t} = d\mathbf{r}/ds \equiv \mathbf{r}' = t^\alpha\mathbf{a}_\alpha$ and the outward unit normal vector $\mathbf{v} = \partial\mathbf{r}/\partial s_v|_{\mathcal{C}} \equiv \mathbf{r}_{,v} = \mathbf{t} \times \mathbf{n} = v^\alpha\mathbf{a}_\alpha$, $v^\alpha = \varepsilon^{\alpha\beta}t_v$, where s_v is the arc length of the coordinate line on $\mathcal{M}$ which is orthogonal to $\mathcal{C}$. Curvature properties of $\mathcal{C}$ are described by the normal curvature $\sigma_t = b_{\alpha\beta}t^\alpha t^\beta$, the geodesic torsion $\tau_t = -b_{\alpha\beta}v^\alpha t^\beta$ and the geodesic curvature $\varkappa_t = t_\alpha v^\alpha|_\beta t^\beta$, where $( )_{|\alpha}$ denotes the covariant surface derivative on $\mathcal{M}$. For other geometric definitions and relations on $\mathcal{M}$ and $\mathcal{C}$ we refer to PIETRASZKIEWICZ [185,190,193].

The deformed configuration $\bar{\mathcal{M}}$ of the surface $\mathcal{M}$ is described by the position vector relative to the same Cartesian frame

$$\bar{\mathbf{r}} = \bar{x}^k(\theta^\alpha)\mathbf{i}_k = \chi(\mathbf{r}) = \mathbf{r} + \mathbf{u} \quad , \tag{2.1}$$

where θ^α is the same surface curvilinear convected (material) coordinates and $\mathbf{u} = u^\alpha \mathbf{a}_\alpha + w\mathbf{n}$ is the displacement field. Geometric quantities and relations, which may be analogously defined on $\mathcal{U}$ and $\bar{\mathcal{C}}$ on a point $\bar{M}$ with the same values of θ^α or s, will be marked by an additional overbar: $\bar{\mathbf{a}}_\alpha$, $\bar{\mathbf{a}}_{\alpha\beta}$, $\bar{a}$, $\bar{\epsilon}^{\alpha\beta}$, $\bar{\mathbf{n}}$, $\bar{b}_{\alpha\beta}$, $\bar{b}$, $\bar{a}^{\alpha\beta}$, $(\quad)_{||\alpha}$, $\bar{s}$, $\bar{t}$, $\bar{t}^\alpha$, $\bar{v}$, $\bar{v}^\alpha$, $\bar{\sigma}_t$, $\bar{\tau}_t$, $\bar{x}_t$ etc.

For the base vectors on $\mathcal{U}$ the following relations hold

$$\bar{\mathbf{a}}_\alpha = \mathbf{G}\mathbf{a}_\alpha = l^\lambda_{\cdot\alpha}\mathbf{a}_\lambda + \varphi_\alpha \mathbf{n} \ ,$$

$$\bar{\mathbf{n}} = \mathbf{G}\mathbf{n} = n^\lambda \mathbf{a}_\lambda + n\mathbf{n} \ , \tag{2.2}$$

where

$$l_{\alpha\beta} = a_{\alpha\beta} + \theta_{\alpha\beta} - \omega_{\alpha\beta} \ , \qquad \varphi_\alpha = w_{,\alpha} + b^\lambda_\alpha u_\lambda \ ,$$

$$\theta_{\alpha\beta} = \frac{1}{2}(u_\alpha|_\beta + u_\beta|_\alpha) - b_{\alpha\beta}w \ , \qquad \omega_{\alpha\beta} = \frac{1}{2}(u_\beta|_\alpha - u_\alpha|_\beta) \ , \tag{2.3}$$

$$n_\alpha = \frac{1}{d}m_\alpha \ , \qquad n = \frac{1}{d}m \ , \qquad d = \sqrt{\frac{\bar{a}}{a}} \ , \tag{2.4}$$

$$m_\mu = \epsilon^{\alpha\beta}\epsilon_{\lambda\mu}\varphi_\alpha l^\lambda_{\cdot\beta} \ , \qquad m = \frac{1}{2}\epsilon^{\alpha\beta}\epsilon_{\lambda\mu}l^\lambda_{\cdot\alpha}l^\mu_{\cdot\beta} \ ,$$

$$\mathbf{G} = \bar{\mathbf{a}}_\alpha \otimes \mathbf{a}^\alpha + \bar{\mathbf{n}} \otimes \mathbf{n} \ , \qquad \mathbf{G}^{-1} = \mathbf{a}_\alpha \otimes \bar{\mathbf{a}}^\alpha + \mathbf{n} \otimes \bar{\mathbf{n}} \ . \tag{2.5}$$

Here $\mathbf{G} \equiv \partial\chi/\partial\mathbf{r}$ is the deformation gradient tensor of the surface $\mathcal{U}$ while $\otimes$ is the tensor product.

The Lagrangian surface strain tensor γ and the tensor of change of surface curvature $\varkappa$ are defined by

$$\gamma = \frac{1}{2}(\mathbf{G}^T\mathbf{G} - 1) = \gamma_{\alpha\beta}\mathbf{a}^\alpha \otimes \mathbf{a}^\beta \ ,$$

$$\varkappa = -(\mathbf{G}^T\bar{b}\mathbf{G} - b) = \varkappa_{\alpha\beta}\mathbf{a}^\alpha \otimes \mathbf{a}^\beta \ , \tag{2.6}$$

$$\gamma_{\alpha\beta} = \frac{1}{2}(\bar{a}_{\alpha\beta} - a_{\alpha\beta}) = \frac{1}{2}(\bar{r}_{,\alpha} \cdot \bar{r}_{,\beta} - a_{\alpha\beta}) \tag{2.7}$$

$$= \frac{1}{2}(1^{\lambda}_{\cdot\alpha}1_{\lambda\beta} + \varphi_{\alpha}\varphi_{\beta} - a_{\alpha\beta}) \ ,$$

$$\varkappa_{\alpha\beta} = -(\bar{b}_{\alpha\beta} - b_{\alpha\beta}) = \tag{2.8}$$

$$= \bar{a}_{\alpha} \cdot \bar{n}_{,\beta} + b_{\alpha\beta} = -\bar{a}_{\alpha,\beta} \cdot \bar{n} + b_{\alpha\beta}$$

$$= 1_{\lambda\alpha}(n^{\lambda}|_{\beta} - b^{\lambda}_{\beta}n) + \varphi_{\alpha}(n_{,\beta} + b^{\lambda}_{\beta}n_{\lambda}) + b_{\alpha\beta} \tag{2.9}$$

$$= -n(\varphi_{\alpha}|_{\beta} + b^{\lambda}_{\beta}1_{\lambda\alpha}) - n_{\lambda}(1^{\lambda}_{\cdot\alpha}|_{\beta} - b^{\lambda}_{\beta}\varphi_{\alpha}) + b_{\alpha\beta} \ , \tag{2.10}$$

where $1 \equiv a_{\alpha} \otimes a^{\alpha} + n \otimes n$ is the metric tensor of the Euclidean space $\mathcal{E}$.

According to [64] the strain-displacement relations $(2.7)_2$ and (2.10) were given first by MUSHTARI [150]. They were applied, among others, in the papers [64,215,115,183,185,291,36,71,72]. The importance of an equivalent representation (2.9) for $\varkappa_{\alpha\beta}$ was recognized only recently by PIETRASZKIEWICZ and SZWABOWICZ [20] and was applied, for example, in [197,198, 267,271,218-222,225,164,165,262, 263].

Note that $\gamma_{\alpha\beta}$ are quadratic polynomials of u_{α}, w and their first surface derivatives while $\varkappa_{\alpha\beta}$ are non-rational functions of u_{α}, w and their first as well as the second surface derivatives. The non-rationality is caused by the presence of the invariant d in the definitions of n_{α} and n appearing in (2.9) and (2.10), where

$$d^2 = \frac{\bar{a}}{a} = 1 + 2\gamma^{\alpha}_{\alpha} + 2(\gamma^{\alpha}_{\alpha}\gamma^{\beta}_{\beta} - \gamma^{\alpha}_{\beta}\gamma^{\beta}_{\alpha}) \ . \tag{2.11}$$

Components of the Lagrangian surface strain measures should satisfy the compatibility conditions originally derived by CHIEN [44] and rederived by GALIMOV [63,65] (with the sign error) and KOITER [115]. We present them in the form given in [185]

$$\varepsilon^{\alpha\beta}\varepsilon^{\lambda\mu}[\varkappa_{\beta\lambda}{}_{|\mu} + \bar{a}^{\varkappa\upsilon}(b_{\varkappa\lambda} - \varkappa_{\varkappa\lambda})\gamma_{\upsilon\beta\mu}] = 0 \ ,$$

(2.12)

$$K\gamma^{\varkappa}{}_{\varkappa} + \varepsilon^{\alpha\beta}\varepsilon^{\lambda\mu}[\gamma_{\alpha\mu}{}_{|\beta\lambda} - b_{\alpha\mu}\varkappa_{\beta\lambda} + \frac{1}{2}(\varkappa_{\alpha\mu}\varkappa_{\beta\lambda} + \bar{a}^{\varkappa\lambda}\gamma_{\varkappa\alpha\mu}\gamma_{\upsilon\beta\lambda})] = 0 \ ,$$

where $K = |b^{\alpha}_{\beta}| = \det b$ is the Gaussian curvature of $\mathcal{M}$ and

$$\gamma_{\upsilon\beta\mu} = \gamma_{\upsilon\beta}{}_{|\mu} + \gamma_{\upsilon\mu}{}_{|\beta} - \gamma_{\beta\mu}{}_{|\upsilon} \ .$$

(2.13)

Alternative form of $(2.12)_2$ is given in [270].

The deformation of the shell lateral boundary element may be described by two vectors

$$u = \bar{r} - r = u_{\upsilon}\boldsymbol{\nu} + u_t t + w n \ ,$$

(2.14)

$$\beta = \bar{n} - n = n_{\upsilon}\boldsymbol{\nu} + n_t t + (n-1)n$$

which are subjected to two geometric constraints

$$\bar{n}\cdot\bar{r}' = 0 \ , \quad \bar{n}\cdot\bar{n} = 1 \ .$$

(2.15)

These constraints imply that among six components of $\bar{r}$ (or u) and $\bar{n}$ (or β) on $\mathcal{C}$ only four are independent: three components of $\bar{r}$ (or u), which determine the translation of the boundary contour $\mathcal{C}$, and a scalar function ϕ which describes the rotational deformation of the shell lateral boundary element. Since the rotational deformation may be described by various means, also various definitions of ϕ may be used in the non-linear theory of shells.

If ϕ is identified with $n_{\upsilon} = \bar{n}\cdot\boldsymbol{\nu}$ then $\bar{n}$ can be expressed entirely in terms of u, u' and n_{υ} , [201]

$$\bar{n} = \frac{1}{c_t^2 + c} [n_{\upsilon}\bar{a}_t \times (\boldsymbol{\nu} \times \bar{a}_t) + \sqrt{\bar{a}_t^2(1-n_{\upsilon}^2) - c_{\upsilon}^2} \ \boldsymbol{\nu} \times \bar{a}_t] \ ,$$

(2.16)

where

$$\bar{a}_t \equiv \bar{r}' = t + u' = c_\nu \nu + c_t t + cn , \quad c_\nu = u'_\nu + \tau_t w - x_t u_t ,$$

$$c_t = 1 + u'_t + x_t u_\nu - \sigma_t w , \quad c = w' + \sigma_t u_t - \tau_t u_\nu ,$$

$$\bar{a}_t = |\bar{a}_t| = \sqrt{1 + 2\gamma_{tt}} , \quad 2\gamma_{tt} = 2\gamma_{\alpha\beta} t^\alpha t^\beta = (\bar{r}')^2 .$$

(2.17)

The relation (2.16) is valid when the rotation of the boundary element does not exceed $\pm \pi/2$. For larger rotations the sign in front of the square root is not unique and may change.

An equivalent description of $\bar{n}$ in terms of displacement derivatives at C is given by

$$\bar{n} = \frac{1}{d}\bar{r}_{,\nu} \times \bar{r}' ,$$

(2.18)

$$\bar{r}_{,\nu} = \nu + u_{,\nu} = \bar{a}_\alpha \nu^\alpha = \frac{1}{\bar{a}_t} (d\bar{\nu} + 2\gamma_{\nu t}\bar{t}) ,$$

$$d^2 = (\bar{r}_{,\nu})^2 (\bar{r}')^2 - (\bar{r}_{,\nu} \cdot \bar{r}')^2 ,$$

(2.19)

$$2\gamma_{\nu t} = 2\gamma_{\alpha\beta} \nu^\alpha t^\beta = \bar{r}_{,\nu} \cdot \bar{r}' .$$

Note that $\bar{r}_{,\nu}$ is not orthogonal to $\tilde{C}$ due to the shear distortion of the surface during deformation.

In what follows we shall use the following transformation

$$\bar{a}_\alpha = \frac{1}{\bar{a}_t}(d\nu_\alpha \bar{\nu} + \bar{a}_{\alpha\beta} t^\beta \bar{t})$$

(2.20)

which holds at the deformed boundary contour $\tilde{C}$.

3. Basic forms of shell equations

The two-dimensional equilibrium equations and the appropriate natural static boundary conditions for the non-linear K-L type theory of shells may be derived in several ways. The usual way is to integrate the corresponding three-dimensional relations of a continua over the shell thickness. This leads to six equilibrium equations and six static boundary conditions, expressed in terms of two-dimensional non-symmetric internal force and couple resultants and the shearing forces. Additional transformations allow then to reduce the relations to three equilibrium equations and four static boundary conditions, which are expressed in terms of two-dimensional symmetric internal force and couple resultants. An alternative direct way is to postulate the two-dimensional virtual work principle compatible with the basic assumptions of the shell theory, from which follow at once the same three equilibrium equations and four static boundary conditions. Internal force and couple resultants are symmetric here by definition, since they appear as coefficients of the symmetric virtual surface strain measures in the invariant virtual work expression. In this report we shall apply the second direct approach, since it leads directly to the final shell equations.

A clear distinction should be made between the set of shell equations written in the Eulerian description and the one written in the Lagrangian description. In the Lagrangian description all quantities and equations are referred to the known, natural basis of the undeformed reference surface. In the Eulerian description they are referred to the natural basis of the deformed surface, the geometry of which is not known in advance. If transformation formulae between deformed and undeformed surface are used to express components of the Eulerian quantities in terms of corresponding Lagrangian ones, then the Eulerian shell equations can be presented in the so called mixed form.

3.1. Eulerian shell equations

Let $\mathcal{A}$ be the reference surface of a thin shell in an equilibrium state, under the surface force $\bar{p} = \bar{p}^\alpha \bar{a}_\alpha + \bar{p}\bar{n}$ and the surface static moment $\bar{h} = \bar{h}^\alpha \bar{a}_\alpha + \bar{h}\bar{n}$, both per unit area of $\mathcal{A}$, as well as under the boundary force $\bar{T} = \bar{T}_\nu \bar{\nu} + \bar{T}_t \bar{t} + \bar{T}\bar{n}$ and the boundary static moment $\bar{H} = \bar{H}_\nu \bar{\nu} + \bar{H}_t \bar{t} + \bar{H}\bar{n}$, both per unit length of $\mathcal{C}$. For an additional virtual displacement field $\delta\bar{u} = \delta\bar{u}_\alpha \bar{a}^\alpha + \delta\bar{w}\bar{n}$, which is subjected to geometric constraints, the internal virtual work performed by the

internal stress and couple resultants on virtual strain measures is equal to the external virtual work, performed by the external surface and boundary loads on appropriate virtual displacement parameters:

$$\iint_{\bar{\mathcal{M}}} (\bar{\mathbf{N}} \cdot \delta\bar{\gamma} + \bar{\mathbf{M}} \cdot \delta\bar{\varkappa})d\bar{A} =$$

$$= \int_{\bar{\mathcal{C}}} (\bar{\mathbf{p}} \cdot \delta\bar{\mathbf{u}} + \bar{\mathbf{h}} \cdot \delta\bar{\beta})d\bar{A} + \int_{\bar{\mathcal{C}}_f} (\bar{\mathbf{T}} \cdot \delta\bar{\mathbf{u}} + \bar{\mathbf{H}} \cdot \delta\bar{\beta})d\bar{s} \quad , \tag{3.1}$$

where $\bar{\mathbf{N}} = \bar{N}^{\alpha\beta}\bar{\mathbf{a}}_\alpha \otimes \bar{\mathbf{a}}_\beta$, $\bar{\mathbf{M}} = \bar{M}^{\alpha\beta}\bar{\mathbf{a}}_\alpha \otimes \bar{\mathbf{a}}_\beta$ are symmetric (Cauchy type) internal stress and couple resultant tensors and

$$\delta\bar{\gamma} = [\tfrac{1}{2}(\delta\bar{u}_{\alpha||\beta} + \delta\bar{u}_{\beta||\alpha}) - \bar{b}_{\alpha\beta}\delta\bar{w}]\bar{\mathbf{a}}^\alpha \otimes \bar{\mathbf{a}}^\beta \quad ,$$

$$\delta\bar{\varkappa} = [-\delta\bar{w}_{||\alpha\beta} - \bar{b}^\lambda_\alpha\delta\bar{u}_{\lambda||\beta} - \bar{b}^\lambda_\beta\delta\bar{u}_{\lambda||\alpha} - \bar{b}^\lambda_{\alpha||\beta}\delta\bar{u}_\lambda + \bar{b}^\lambda_\alpha\bar{b}_{\lambda\beta}\delta\bar{w}]\bar{\mathbf{a}}^\alpha \otimes \bar{\mathbf{a}}^\beta \quad , \tag{3.2}$$

$$\delta\bar{\beta} = -(\delta\bar{w}_{,\alpha} + \bar{b}^\lambda_\alpha\delta\bar{u}_\lambda)\bar{\mathbf{a}}^\alpha \quad .$$

After elementary transformations (3.1) takes the form

$$-\iint_{\bar{\mathcal{M}}} (\bar{\mathbf{N}}^\beta_{||\beta} + \bar{\mathbf{p}}) \cdot \delta\bar{u}d\bar{A} + \int_{\bar{\mathcal{C}}_f} [(\bar{\mathbf{P}} - \bar{\mathbf{P}}^*) \cdot \delta\bar{\mathbf{u}} + (\bar{M} - \bar{M}^*)\delta\bar{\beta}_\nu]d\bar{s} + \sum_j (\bar{\mathbf{F}} - \bar{\mathbf{F}}^*_j) \cdot \delta\bar{\mathbf{u}}_j = 0 \quad , \tag{3.3}$$

$$\bar{\mathbf{N}}^\beta = (\bar{N}^{\alpha\beta} - \bar{b}^\alpha_\lambda\bar{M}^{\lambda\beta})\bar{\mathbf{a}}_\alpha + (\bar{M}^{\alpha\beta}_{||\alpha} + \bar{h}^\beta)\bar{\mathbf{n}} \quad ,$$

$$\bar{\mathbf{P}} = \bar{\mathbf{N}}^\beta\bar{\nu}_\beta + \frac{d}{d\bar{s}}\bar{\mathbf{F}} \quad , \qquad \bar{\mathbf{P}}^* = \bar{\mathbf{T}} + \frac{d}{d\bar{s}}\bar{\mathbf{F}}^* \quad ,$$

$$\bar{M} = \bar{M}^{\alpha\beta}\bar{\nu}_\alpha\bar{\nu}_\beta \quad , \qquad \bar{M}^* = \bar{H}_\nu \quad , \tag{3.4}$$

$$\bar{\mathbf{F}} = \bar{M}^{\alpha\beta}\bar{\nu}_\alpha\bar{t}_\beta\bar{\mathbf{n}} \quad , \qquad \bar{\mathbf{F}}^* = \bar{H}_t\bar{\mathbf{n}} \quad ,$$

$$\bar{F}_j = \bar{F}(\bar{s}_j + 0) - \bar{F}(\bar{s}_j - 0) \quad , \qquad \delta\bar{\beta}_\nu = \delta\bar{\beta}\cdot\bar{\nu} \quad , \qquad \delta\bar{u}_j = \delta\bar{u}(\bar{s}_j) \quad .$$

For arbitrary $\delta\bar{u}$ on $\mathscr{A}$ and $\delta\bar{u}$, $\delta\bar{\beta}_\nu$ and $\delta\bar{u}_j$ on $\bar{\mathcal{C}}_f$ from (3.3) follow Eulerian equilibrium equations and corresponding static boundary conditions for the free edge [197]

$$\bar{N}^\beta||_\beta + \bar{p} = 0 \quad \text{in } \mathscr{A} \ ,$$

$$\bar{P} = \bar{P}^* \ , \ \bar{M} = \bar{M}^* \quad \text{on} \quad \bar{\mathcal{C}}_f \ , \tag{3.5}$$

$$\bar{F}_j = \bar{F}_j^* \quad \text{at each corner} \quad \bar{M}_j \ \epsilon \ \bar{\mathcal{C}}_f \ .$$

The virtual rotation $\delta\bar{\beta}_\nu$ on $\bar{\mathcal{C}}_f$, appearing in the boundary line integral of (3.3), may also be given in alternative but equivalent forms

$$\delta\bar{\beta}_\nu = \bar{\nu}\cdot\delta\bar{n} = -\ \bar{n}\cdot\delta\bar{\nu} \ ,$$

$$= -\ (\delta\bar{w}_{,\alpha} + \bar{b}_\alpha^\lambda \delta\bar{u}_\lambda)\bar{\nu}^\alpha = -\ (\bar{n}\cdot\delta\bar{u})_{,\bar{\nu}} + \bar{n}_{,\bar{\nu}}\cdot\delta\bar{u} \ , \tag{3.6}$$

$$= -\ \bar{n}\cdot(\delta\bar{u})_{,\bar{\nu}} \quad .$$

Here $(\ )_{,\bar{\nu}} \equiv \partial(\ )/\partial\bar{x}_\nu|_{\bar{\mathcal{C}}}$ such that $\bar{r}_{,\bar{\nu}} = \bar{\nu}$ on $\bar{\mathcal{C}}$, where $\bar{x}_\nu$ is the arc length of the coordinate line on $\mathscr{A}$ which is orthogonal to $\bar{\mathcal{C}}$.

Using $(3.6)_2$ the line integral of (3.3) may also be transformed into the alternative form

$$\int_{\bar{\mathcal{C}}_f} [(\bar{P}_1 - \bar{P}_1^*)\cdot\delta\bar{u} - (\bar{M} - \bar{M}^*)(\bar{n}\cdot\delta\bar{u})_{,\bar{\nu}}]d\bar{s} \ , \tag{3.7}$$

$$\bar{P}_1 = \bar{N}^\beta\bar{\nu}_\beta + \frac{d}{d\bar{s}}\ \bar{F} + \bar{M}\bar{n}_{,\bar{\nu}} \ , \tag{3.8}$$

$$\bar{P}_1^* = \bar{T} + \frac{d}{d\bar{s}}\ \bar{F}^* + \bar{M}^*\bar{n}_{,\bar{\nu}} \ ,$$

which leads to a modified static boundary condition $\bar{P}_1 = \bar{P}_1^*$ on $\bar{\mathcal{C}}_f$ in $(3.5)_2$.

Tensor form of the Eulerian equilibrium equations, but expressed in terms of non-symmetric stress and couple resultant tensors, was given first independently by LURIE [136] and by SYNGE and CHIEN [266] while GALIMOV [63] derived static boundary conditions for the smooth $\bar{\mathcal{C}}_f$. In terms of symmetric stress and couple resultant tensors and for smooth $\bar{\mathcal{C}}_f$ the relations $(3.5)_{1,2}$ were given first by GALIMOV [64] (cf. also [71]). The equilibrium equations $(3.5)_1$ were rederived also by SANDERS [215] and KOITER [115]. The final form of (3.5) without $\bar{h}$ was given by the author [185]. The modified static boundary conditions on $\bar{\mathcal{C}}_f$ resulting from (3.7) were given first by KOITER [115] and rederived by ZUBOV [304]. It was noted already by LURIE [136] that the structure of Eulerian shell equations (3.5) is exactly the same as the one of the classical linear theory of shells, only all the quantities are referred now to the geometry of the deformed reference surface $\mathcal{M}$ and of its boundary contour $\bar{\mathcal{C}}$.

As it has been mentioned above, the geometry of $\mathcal{M}$ and $\bar{\mathcal{C}}$ is usually not known in advance and should be determined as an outcome of the solution of the non-linear shell problem. As a result, the simple Eulerian shell equations (3.5) can not be used directly to analyse the shell problems, but they can serve as the basis for derivation of other mixed forms of shell equations. The virtual displacement parameters $\delta\bar{u}_\alpha = \bar{a}_\alpha \cdot \delta\bar{u}$, $\delta\bar{w} = \bar{n}\cdot\delta\bar{u}$ and $\delta\bar{\beta}_\nu = \bar{\nu}\cdot\delta\bar{n}$ should not be identified here with variations of displacement and rotation components, since the respective bases $\bar{a}_\alpha$, $\bar{n}$ and $\bar{\nu}$, $\bar{t}$, $\bar{n}$ of $\mathcal{M}$ and $\bar{\mathcal{C}}$ are themselves subjected to the variation. In particular, $\delta\bar{\beta}_\nu$ should not be identified with the variation of $\bar{\nu}\cdot\beta$. This is the reason why no work-conjugate geometric boundary conditions expressed in terms of displacement parameters can be associated with the Eulerian shell equations (3.5).

3.2. Lagrangian shell equations

Usually only the undeformed configuration of the shell is the one which is know in advance, while the deformed configuration is the one which should be determined in the process of solution. Therefore, it is desirable to construct the equilibrium equations and corresponding boundary and corner conditions, which are expressed entirely in the geometry of $\mathcal{M}$ and $\mathcal{C}$. Such Lagrangian shell equations can be derived with the help of transformation rules between deformed and

undeformed surface geometries [185]

$$d\bar{A} = \sqrt{\frac{\bar{a}}{a}}\, dA \quad , \qquad d\bar{s} = \bar{a}_t d\bar{s} \quad ,$$

$$\bar{\nu}_\beta d\bar{s} = \sqrt{\frac{\bar{a}}{a}}\, \nu_\beta d\bar{s}, \qquad \bar{t}_\beta d\bar{s} = (\delta^\alpha_\beta + 2\gamma^\alpha_\beta) t_\alpha ds \quad ,$$

$$\bar{\nu}^\beta d\bar{s} = \sqrt{\frac{\bar{a}}{a}}\, (\delta^\beta_\alpha + 2\varepsilon_{\alpha\lambda}\varepsilon^{\beta\mu}\gamma^\lambda_\mu)\nu^\alpha ds \quad , \qquad \bar{t}^\beta d\bar{s} = t^\beta ds.$$

$$\tag{3.9}$$

Let us introduce the symmetric (2nd Piola-Kirchhoff type) internal stress and couple resultant tensors $N = N^{\alpha\beta} a_\alpha \otimes a_\beta$, $M = M^{\alpha\beta} a_\alpha \otimes a_\beta$, the Lagrangian surface force $p = p^\alpha a_\alpha + pn$ and the surface static moment $h = h^\alpha a_\alpha + hn$, both per unit area of $\mathscr{M}$, as well as the Lagrangian boundary force $T = T_\nu \nu + T_t t + Tn$ and the boundary static moment $H = H_\nu \nu + H_t t + Hn$, both per unit length of $\mathscr{C}$, by the following relations

$$\bar{N} = \frac{1}{d}\, GNG^T \quad , \qquad \bar{M} = \frac{1}{d}\, GMG^T \quad ,$$

$$\tag{3.10}$$

$$\bar{p} = \frac{1}{d}\, p \quad , \qquad \bar{h} = \frac{1}{d}\, h \quad , \qquad \bar{T} = \frac{1}{\bar{a}_t}\, T \quad , \qquad \bar{H} = \frac{1}{\bar{a}_t}\, H \quad .$$

Let us also note that the virtual strain measures in (3.1) are transformed according to

$$\delta\bar{\gamma} = G^{-T}\delta\gamma G^{-1} \quad , \qquad \delta\bar{\varkappa} = G^{-T}\delta\varkappa G^{-1} .$$

$$\tag{3.11}$$

With the help of (3.9), (3.10) and (3.11) the principle of virtual work (3.1) is transformed into the Lagrangian principle of virtual displacements

$$\iint_{\mathscr{M}} (N\cdot\delta\gamma + M\cdot\delta\varkappa)dA =$$

$$\tag{3.12}$$

$$= \iint_{\mathscr{M}} (p\cdot\delta u + h\cdot\delta\beta)dA + \int_{\mathscr{C}_f} (T\cdot\delta u + H\cdot\delta\beta)ds \quad ,$$

where now

$$\delta u = \delta u_\alpha a^\alpha + \delta w n \quad , \qquad \delta\beta = \delta n_\nu \nu + \delta n_t t + \delta n n \quad ,$$

$$\delta\gamma = \delta\gamma_{\alpha\beta} a^\alpha \otimes a^\beta \quad , \qquad \delta\varkappa = \delta\varkappa_{\alpha\beta} a^\alpha \otimes a^\beta \quad , \tag{3.13}$$

$$\delta\gamma_{\alpha\beta} = \frac{1}{2}(\bar{a}_\alpha \cdot \delta u_{,\beta} + \bar{a}_\beta \cdot \delta u_{,\alpha}) \quad ,$$

$$\delta\varkappa_{\alpha\beta} = \frac{1}{2}(\bar{n}_{,\alpha} \cdot \delta u_{,\beta} + \bar{n}_{,\beta} \cdot \delta u_{,\alpha} + \bar{a}_\alpha \cdot \delta n_{,\beta} + \bar{a}_\beta \cdot \delta\bar{n}_{,\alpha}) \tag{3.14}$$

are variations of the displacement and strain measures, since the bases a_α, n and ν, t, n are fixed and not subjected to the variation during the shell deformation.

After involved transformations given in [197], and taking into account that $\delta\beta = \delta\bar{n} = - (\bar{a}^\beta \otimes \delta u_{,\beta})\bar{n}$ in $\mathcal{M}$, the principle (3.12) can be transformed into

$$- \iint\limits_{\mathcal{M}} (T^\beta|_\beta + p) \cdot \delta u \, dA +$$

$$+ \int\limits_{\mathcal{C}_f} [(P - P^*) \cdot \delta u + (M - M^*)\delta n_\nu] ds + \sum_j (F_j - F_j^*) \cdot \delta u_j \quad , \tag{3.15}$$

where now

$$T^\beta = T^{\lambda\beta} a_\lambda + T^\beta n + (h \cdot \bar{a}^\beta)\bar{n}$$

$$= N^{\alpha\beta}\bar{a}_\alpha + M^{\alpha\beta}\bar{n}_{,\alpha} + [(M^{\varkappa\rho}\bar{a}_\varkappa)|_\rho \cdot \bar{a}^\beta]\bar{n} + (h \cdot \bar{a}^\beta)\bar{n} \quad , \tag{3.16}$$

$$P = T^\beta \nu_\beta + F' \quad , \qquad P^* = T + F^{*'} \quad ,$$

$$F = - \frac{1}{a_\nu} [(\bar{n} \times \bar{a}_\alpha) \cdot \nu]M^{\alpha\beta}\nu_\beta\bar{n} \quad , \qquad M = \frac{1}{a_\nu} (\bar{n} \times \bar{a}_\alpha) \cdot \bar{a}_t M^{\alpha\beta}\nu_\beta \quad , \tag{3.17}$$

$$F^* = - \frac{1}{a_\nu} [(\bar{n} \times H) \cdot \nu]\bar{n} \quad , \qquad M^* = \frac{1}{a_\nu} (\bar{n} \times H) \cdot \bar{a}_t \quad , \quad a_\nu = (\bar{a}_t \times \bar{n}) \cdot \nu \quad ,$$

$$F_j = F(s_j + 0) - F(s_j - 0) \quad , \quad \delta u_j = \delta u(s_j) \quad .$$

Here $\bar{a}_\alpha$, $\bar{n}$ are understood to be expressed in terms of a_α, n and u, n_ν , what gives

$$T^{\lambda\beta} = 1^\lambda_{\cdot\alpha}(N^{\alpha\beta} - \bar{b}^\alpha_\lambda M^{\lambda\beta}) + n^\lambda[M^{\alpha\beta}|_\alpha + \bar{a}^{\beta\varkappa}(2\gamma_{\varkappa\lambda}|_\mu - \gamma_{\lambda\mu}|_\varkappa)M^{\lambda\mu}] \quad ,$$
$$(3.18)$$

$$T^\beta = \varphi_\alpha(N^{\alpha\beta} - \bar{b}^\alpha_\lambda M^{\alpha\beta}) + n[M^{\alpha\beta}|_\alpha + \bar{a}^{\beta\varkappa}(2\gamma_{\varkappa\lambda}|_\mu - \gamma_{\lambda\mu}|_\varkappa)M^{\lambda\mu}] \quad ,$$

$$F = (g_\nu R_{t\nu} + r_\nu R_\nu)\nu + (g_t R_{t\nu} + r_t R_\nu)t + (gR_{t\nu} + rR_\nu)n \quad ,$$
$$(3.19)$$

$$F^* = (g_\nu H_t + r_\nu H)\nu + (g_t H_t + r_t H)t + (gH_t + rH)n \quad ,$$

$$M = R_{\nu\nu} + fR_{t\nu} + kR_\nu \quad , \qquad M^* = H_\nu + fH_t + kH \quad ,$$
$$(3.20)$$

$$R_{\nu\nu} = \nu^\lambda 1_{\lambda\alpha} M^{\alpha\beta} \nu_\beta \quad , \qquad R_{t\nu} = t^\lambda 1_{\lambda\alpha} M^{\alpha\beta} \nu_\beta \quad , \quad R_\nu = \varphi_\alpha M^{\alpha\beta} \nu_\beta \quad ,$$

where g_ν, g_t, g, r_ν, r_t, r are complex functions of u, n_ν given in [197].

An alternative representation for $T^{\lambda\beta}$, T^β in $(3.16)_1$ can also be derived [197]

$$T^{\lambda\beta} = [N^{\alpha\beta} + \bar{a}^{\alpha\beta}(A_\mu n^\mu + An)]1^\lambda_{\cdot\alpha} + M^{\alpha\beta}(n^\lambda|_\alpha - b^\lambda_\alpha n) +$$

$$+ \frac{1}{d} \varepsilon^{\alpha\beta}\varepsilon^{\lambda\mu}(A1_{\mu\alpha} - A_\mu\varphi_\alpha) \quad ,$$
$$(3.21)$$

$$T^\beta = [N^{\alpha\beta} + \bar{a}^{\alpha\beta}(A_\mu n^\mu + An)]\varphi_\alpha + M^{\alpha\beta}(n_{,\alpha} + b^\lambda_\alpha n_\lambda) +$$

$$+ \frac{1}{d} \varepsilon^{\alpha\beta}\varepsilon^{\lambda\mu}A_\mu 1_{\lambda\alpha} \quad ,$$

$$\bar{a}^{\alpha\beta} = \frac{1}{d^2} [(1+2\gamma_\varkappa^\varkappa)a^{\alpha\beta} - 2\gamma^{\alpha\beta}] \ ,$$

$$\tag{3.22}$$

$$A_\mu = (M^{\varkappa\rho}1_{\mu\varkappa})|_\rho - M^{\varkappa\rho}\varphi_\varkappa b_{\mu\rho} \ , \qquad A = (M^{\varkappa\rho}\varphi_\varkappa)|_\rho + M^{\varkappa\rho}1_{\gamma\varkappa}b_\rho^\gamma \ .$$

For arbitrary δu in $\mathcal{M}$ and δu, δn_ν and δu_j on $\mathcal{C}_f$ from (3.15) follow now entirely Lagrangian equilibrium equations and corresponding static boundary and corner conditions

$$T^\beta|_\beta + p = 0 \quad \text{in } \mathcal{M} ,$$

$$P = P^* \ , \ M = M^* \quad \text{on } \mathcal{C}_f \ ,$$

$$\tag{3.23}$$

$$F_j = F_j^* \quad \text{at each corner } M_j \in \mathcal{C}_f \ .$$

Corresponding work-conjugate geometric boundary conditions are

$$u = u^* \ , \ n_\nu = n_\nu^* \quad \text{on } \mathcal{C}_u \ ,$$

$$\tag{3.24}$$

$$u_i = u_i^* \quad \text{at each corner } M_i \in \mathcal{C}_u \ .$$

The equivalent entirely Lagrangian shell equations (3.23) and (3.24) (without h) were first derived by PIETRASZKIEWICZ and SZWABOWICZ [201] using modified tensor of change of curvature $\chi_{\alpha\beta}$, which by definition is a third-degree polynomial in displacements and their surface derivatives (see also [267,271]). Alternative equivalent formulations, in terms of the modified tensor of change of curvature proposed by BUDIANSKY [36], were given in [218,97]. In terms of $\varkappa_{\alpha\beta}$ the Lagrangian shell relations were derived by the author [197,198] and in [221,223,96].

Let us note that already GALIMOV [63] proposed a version of Lagrangian shell equations by transforming the final Eulerian vector relations into the unde-formed configuration and resolving them in components with respect to the undeformed basis. Under such a transformation the fourth static boundary condi-tion for the couple still remained to be defined with respect to the tangent of

the deformed boundary contour $\tilde{C}$. In [67] it was shown that such a condition appears as a multiplier of the kinematic parameter $\bar{v} \cdot \delta \bar{n}$ in the transformed principle of virtual work (cf. (3.3) and $(3.6)_1$). In order to construct the corresponding geometric boundary conditions, the parameter called "rotation" was defined formally as $\Omega = \int \bar{v} \cdot \delta \bar{n}$, such that $\delta\Omega = \bar{v} \cdot \delta \bar{n}$, and was extensively used in [71,72] for the construction of variational principles. But it is obvious that so defined Ω can not describe the total rotation of the boundary for an arbitrary deformation of the shell and Galimov himself was apparently aware that this representation is not consistent (see discussion on p. 14 of [67]). Various forms of Lagrangian equilibrium equations, but without boundary conditions, were also proposed by GLOCKNER [242], SANDERS [215] and BUDIANSKY [36]. PIETRASZKIEWICZ [183] derived the complete set of Lagrangian shell equations with the fourth static boundary condition compatible with the kinematic parameter $(\bar{n} \cdot \delta u)_{,\nu}$ and in [193] with the kinematic parameter $\bar{a}_t \cdot \delta\Omega_t$, where Ω_t is the total finite rotation vector, but the corresponding work-conjugate geometric boundary conditions were not constructed. In the section 4.4 below we shall prove that the kinematic parameters $\bar{v} \cdot \delta \bar{n}$, $(\bar{n} \cdot \delta u)_{,\nu}$ and $\bar{a}_t \cdot \delta\Omega_t$ are not integrable, in general, i. e. there exists no scalar function such that its variation would give us the kinematic parameters $\bar{v} \cdot \delta \bar{n}$, $(\bar{n} \cdot \delta u)_{,\nu}$ or $\bar{a}_t \cdot \delta\Omega_t$, even multiplied by another scalar function.

The Eulerian and Lagrangian shell equations are equivalent within the basic assumptions of the K-L type theory of shells. However, the procedures allowing for a reduction to four the number of independent boundary conditions are different in both descriptions. As a result, numerical values of the Eulerian static boundary parameters $\bar{F}$, $\bar{F}_j$ and $\bar{M}$ may differ, in general, from the numerical values of the corresponding Lagrangian static boundary parameters F , F_j and M , see [197].

3.3. Mixed shell equations

For some problems it is convenient to express the component form of the Eulerian shell equations (3.5), written in the basis $\bar{a}_\alpha$, $\bar{n}$ and $\bar{v},\bar{t},\bar{n}$ of the deformed reference surface, in terms of components of vectors and tensors measured with respect to the undeformed surface geometry.

Let us introduce the symmetric (Kirchhoff type) internal stress and couple resultant tensors $N_K = N^{\alpha\beta}\bar{a}_\alpha \otimes \bar{a}_\beta$, $M_K = M^{\alpha\beta}\bar{a}_\alpha \otimes \bar{a}_\beta$ related to the Eulerian and Lagrangian resultant tensors by

$$N_K = d\bar{N} = GNG^T , \qquad M_K = d\bar{M} = GMG^T ,$$

$$(3.25)$$

$$N^{\alpha\beta} = d\bar{N}^{\alpha\beta} , \qquad M^{\alpha\beta} = d\bar{M}^{\alpha\beta} .$$

Note that in the convected system of coordinates used here components of the Kirchhoff type resultant tensors N_K, M_K in the deformed basis $\bar{a}_\alpha \otimes \bar{a}_\beta$ are exactly the same as components of the 2nd Piola-Kirchhoff type resultant tensors N, M in the undeformed basis $a_\alpha \otimes a_\beta$.

Let p, h, T and H defined by (3.10) are supposed to be given through their components in the deformed basis

$$p = q^\alpha \bar{a}_\alpha + q\bar{n} , \qquad h = k^\alpha \bar{a}_\alpha + k\bar{n} .$$

$$(3.26)$$

$$T = Q_\nu \bar{\nu} + Q_t \bar{t} + Q\bar{n} , \qquad H = K_\nu \bar{\nu} + K_t \bar{t} + K\bar{n} .$$

Then it follows from (3.25) and (3.10)$_2$ that the virtual work principle (3.1) can be transformed into

$$\iint_{\mathscr{M}} (N_K \cdot \delta\bar{\gamma} + M_K \cdot \delta\bar{\varkappa})dA =$$

$$(3.27)$$

$$= \iint_{\mathscr{M}} (P \cdot \delta\bar{u} + h \cdot \delta\bar{\beta})dA + \int_{\mathcal{C}_f} (T \cdot \delta\bar{u} + H \cdot \delta\bar{\beta})ds .$$

After additional transformations we obtain

$$- \iint_{\mathscr{M}} (N^\beta|_\beta + p) \cdot \delta\bar{u}dA - \sum_j (H_j - H_j^*)\bar{n}_j \cdot \delta\bar{u}_j +$$

$$(3.28)$$

$$+ \int_{\mathcal{C}_f} \{[N^\beta \nu_\beta - (H\bar{n})' - T + (H^*\bar{n})'] \cdot \delta\bar{u} + (G - G^*)\bar{\nu} \cdot \delta\bar{n}\}ds = 0 ,$$

where now

$$N^{\beta} = (N^{\alpha\beta} - \bar{b}^{\alpha}_{\lambda}M^{\lambda\beta})\bar{a}_{\alpha} + (M^{\alpha\beta}|_{\alpha} + \bar{a}^{\beta\varkappa}\gamma_{\varkappa\lambda\mu}M^{\lambda\mu})\bar{n} + k^{\beta}\bar{n} \ ,$$

$$G = \frac{1}{\bar{a}_t} \sqrt{\frac{\bar{a}}{a}} \, M^{\alpha\beta}\nu_{\alpha}\nu_{\beta} \ , \qquad G^* = H\cdot\bar{\nu} \ , \tag{3.29}$$

$$H = -\frac{1}{\bar{a}^2_t} M^{\alpha\beta}\bar{a}_{\alpha\lambda}t^{\lambda}\nu_{\beta} \ , \qquad H^* = -\frac{1}{\bar{a}_t} H\cdot\bar{t} \ .$$

For arbitrary $\delta\bar{u}$, $\bar{\nu}\cdot\delta\bar{n}$ and $\delta\bar{u}_j$ from (3.28) follow the mixed equilibrium equations and corresponding static boundary conditions for the free edge

$$N^{\beta}|_{\beta} + p = 0 \quad \text{in} \ \mathcal{M} \ ,$$

$$N^{\beta}\nu_{\beta} - (H\bar{n})' = T - (H^*\bar{n})' , \quad G = G^* \text{ on } \mathcal{C}_f \ , \tag{3.30}$$

$$H_j\bar{n}_j = H^*_j\bar{n}_j \quad \text{at each corner} \ M_j \ \varepsilon \ \mathcal{C}_f \ .$$

Since the mixed shell equations (3.30) are referred to the deformed basis $\bar{a}_{\alpha},\bar{n}$, their component form is

$$(N^{\alpha\beta} - \bar{b}^{\alpha}_{\varkappa}M^{\varkappa\beta})|_{\beta} + \bar{a}^{\alpha\varkappa}\gamma_{\varkappa\lambda\beta}(N^{\lambda\beta} - \bar{b}^{\lambda}_{\mu}M^{\mu\beta}) -$$

$$- \bar{b}^{\alpha}_{\beta}(M^{\lambda\beta}|_{\lambda} + \bar{a}^{\beta\varkappa}\gamma_{\varkappa\lambda\mu}M^{\lambda\mu}) + q^{\alpha} - \bar{b}^{\alpha}_{\beta}k^{\beta} = 0 \ , \tag{3.31}$$

$$M^{\alpha\beta}|_{\alpha\beta} + (\bar{a}^{\beta\varkappa}\gamma_{\varkappa\lambda\mu}M^{\lambda\mu})|_{\beta} + \bar{b}_{\alpha\beta}(N^{\alpha\beta} - \bar{b}^{\alpha}_{\varkappa}M^{\varkappa\beta}) + q + k^{\beta}|_{\beta} = 0 \ .$$

The mixed shell equations (3.30) and their component form (3.31) were given first by GALIMOV [64,67,70] and rederived by DANIELSON [49]. Since only two-dimensional stress and strain measures appear explicity in (3.31), these equilibrium equations are particularly useful if the shell problems are solved in the intrinsic way, cf. [120,185,187].

3.4. Constitutive equations

Within the first-approximation theory of thin isotropic elastic shells the strain energy density, per unit area of $\mathcal{M}$, is given by the sum of two quadratic functions describing the stretching and the bending energies of the shell reference surface. This conclusion was given already by ARON [13] and LOVE [135] within the classical linear theory of shells. The accuracy of such an approximation was discussed, among others, by BASSET [27], LAMB [123], NOVOZHILOV and FINKELSHTEIN [177], GOLDENVEIZER [79-81], KOITER [113,118], DANIELSON [50], KRÄTZIG [122] and RYCHTER [212]. Within the geometrically non-linear theory, according to JOHN [101] and KOITER [115], the strain energy density of the shell is given by

$$\Sigma = \frac{h}{2} H^{\alpha\beta\lambda\mu} (\gamma_{\alpha\beta}\gamma_{\lambda\mu} + \frac{h^2}{12} \varkappa_{\alpha\beta}\varkappa_{\lambda\mu}) + O(Eh\eta^2\theta^2) \quad , \tag{3.32}$$

$$H^{\alpha\beta\lambda\mu} = \frac{E}{2(1+\nu)} (a^{\alpha\lambda}a^{\beta\mu} + a^{\alpha\mu}a^{\beta\lambda} + \frac{2\nu}{1-\nu} a^{\alpha\beta}a^{\lambda\mu}) \quad ,$$

where E is the Young's modulus and ν is the Poisson's ratio of the linearly-elastic material. The error of Σ at any point of $\mathcal{M}$ is expressed in terms of the small parameter θ defined in [101,115,119] to be

$$\theta = \max (\frac{h}{b} , \frac{h}{L} , \frac{h}{l} , \sqrt{\frac{h}{R}} , \sqrt{\eta}) \quad , \tag{3.33}$$

where b is the distance of the point from the lateral shell boundary and other quantities are defined in the Introduction.

The modified elasticity tensor $H^{\alpha\beta\lambda\mu}$ defined by $(3.32)_2$ takes implicity into account the change of the shell thickness during deformation according to the plane stress state in the shell, cf. [189].

Differentiating $(3.32)_1$ with respect to the strain measures we obtain the constitutive equations

$$N^{\alpha\beta} = \frac{\partial\Sigma}{\partial\gamma_{\alpha\beta}} = \frac{Eh}{1-\nu^2} [(1-\nu)\gamma^{\alpha\beta} + \nu a^{\alpha\beta}\gamma_\varkappa^\varkappa] + O(Eh^2\eta\theta^2) \quad , \tag{3.34}$$

$$M^{\alpha\beta} = \frac{\partial \Sigma}{\partial \varkappa_{\alpha\beta}} = \frac{Eh^3}{12(1-\nu^2)} [(1-\nu)\varkappa^{\alpha\beta} + \nu a^{\alpha\beta}\varkappa_\varkappa^\varkappa] + 0(Eh^2\eta\theta^2) \ .$$

Inversion of (3.34) leads to

$$\gamma_{\alpha\beta} = \frac{1}{Eh} [(1+\nu)N_{\alpha\beta} - \nu a_{\alpha\beta}N_\lambda^\lambda] + 0(\eta\theta^2) \ ,$$

$$\varkappa_{\alpha\beta} = \frac{12}{Eh^2} [(1+\nu)M_{\alpha\beta} - \nu a_{\alpha\beta}M_\lambda^\lambda] + 0(\frac{\eta\theta^2}{h}) \ . \tag{3.35}$$

In some variational principles it is convenient to apply the Legendre transformation

$$\Sigma^C(N^{\alpha\beta},M^{\alpha\beta}) = N^{\alpha\beta}\gamma_{\alpha\beta} + M^{\alpha\beta}\varkappa_{\alpha\beta} - \Sigma(\gamma_{\alpha\beta},\varkappa_{\alpha\beta}) \ , \tag{3.36}$$

from which follow the complementary energy density of the shell

$$\Sigma^C = \frac{1}{h} E_{\alpha\beta\lambda\mu} (N^{\alpha\beta}N^{\lambda\mu} + \frac{12}{h^2} M^{\alpha\beta}M^{\lambda\mu}) + 0(Eh\eta^2\theta^2) \ ,$$

$$E_{\alpha\beta\lambda\mu} = \frac{1+\nu}{2E} (a_{\alpha\lambda}a_{\beta\mu} + a_{\alpha\mu}a_{\beta\lambda} - \frac{2\nu}{1+\nu} a_{\alpha\beta}a_{\lambda\mu}) \ . \tag{3.37}$$

Now the inverse constitutive equations may also be defined in terms of Σ^C by

$$\gamma_{\alpha\beta} = \frac{\partial \Sigma^C}{\partial N^{\alpha\beta}} \ , \quad \varkappa_{\alpha\beta} = \frac{\partial \Sigma^C}{\partial M^{\alpha\beta}} \ . \tag{3.38}$$

It is worthwhile to note, that while the equilibrium equations and compatibility conditions are exact on the reference surface (although incomplete from the three-dimensional point of view) the constitutive equations are always approximate. In general, the energy densities Σ and Σ^C are infinite series of the two-dimensional strain and stress measures, respectively, and have to be

consistently approximated for any type of two-dimensional theory of shells.

Within the error already introduced into Σ in $(3.32)_1$ by the simplifying assumptions of the first-approximation theory of shells, some alternative definitions for the two-dimensional measure of change of curvature may be used, for example

$$\hat{\rho}_{\alpha\beta} = \varkappa_{\alpha\beta} + \frac{1}{2} (b_\alpha^\lambda \gamma_{\lambda\beta} + b_\beta^\lambda \gamma_{\lambda\alpha}) \ ,$$

$$K_{\alpha\beta} = - (d\bar{b}_{\alpha\beta} - b_{\alpha\beta}) + b_{\alpha\beta}\gamma_\varkappa^\varkappa + \frac{1}{2} (b_\alpha^\lambda \gamma_{\lambda\beta} + b_\beta^\lambda \gamma_{\lambda\alpha}) \ , \tag{3.39}$$

$$\chi_{\alpha\beta} = - (d\bar{b}_{\alpha\beta} - b_{\alpha\beta}) + b_{\alpha\beta}\gamma_\varkappa^\varkappa \ .$$

Each of the measures (3.39) can be expressed in terms of displacements either using the formula (2.9) or (2.10). The measure $-\hat{\rho}_{\alpha\beta}$ with (2.10) was introduced by KOITER [115] and used in [183,187,190,97]. Without displacemental representation the measure $-\hat{\rho}_{\alpha\beta}$ was applied by KOITER and SIMMONDS [120] to derive the canonical intrinsic shell equations (cf. [187,190]). The measure $K_{\alpha\beta}$ with (2.10) was introduced by BUDIANSKY [36], while with (2.9) if was applied in [218,225]. The measure $\chi_{\alpha\beta}$ with (2.9) was introduced by PIETRASZKIEWICZ and SZWABOWICZ [201] and then applied in [202,267,268,271,164,165,262].

The main advantage of using $K_{\alpha\beta}$ and $\chi_{\alpha\beta}$ is that they are third-degree polynomials in displacements and their first and second derivatives, while $\hat{\rho}_{\alpha\beta}$ and $K_{\alpha\beta}$ when linearized reduce to the measure of change of curvature supposed to be the best one for the linear theory of shells according to BUDIANSKY and SANDERS [37]. The disadvantage of using the modified measures (3.39) in the general theory of shells is that their definitions are not invariant under the change of the reference configuration. With γ and $\varkappa$ we can always associate the equivalent Eulerian strain measures $\bar{\gamma}$ and $\bar{\varkappa}$ defined by [185]

$$\bar{\gamma} = \frac{1}{2} (1 - G^{-T}G^{-1}) \ , \quad \bar{\varkappa} = - (\bar{b} - G^{-T}bG^{-1}) \tag{3.40}$$

which satisfy the following transformation rules

$$\gamma = G^T \bar{\gamma} G \ , \qquad \varkappa = G^T \bar{\varkappa} G \ . \tag{3.41}$$

No equivalent exact definitions of the modified measures (3.39) in the Eulerian description can be given which would satisfy the transformation rule (3.41). This becomes an important disadvantage of the modified measures (3.39) when exact superposition of two arbitrary deformations is discussed, what is necessary in correct incremental analysis of the highly non-linear shell problems [141,197]. An alternative symmetric measure $\rho_{\alpha\beta}$ for the change of curvature, which is free from such disadvantages and when linearized reduces to the best measure of the linear shell theory, was introduced by ALUMÄE [8] and will be used in the chapters 5 and 6 of this report.

4. Shell equations in terms of displacements

The majority of non-linear shell problems discussed in the literature has been formulated and solved in terms of displacements as basic independent field variables. The primary advantage of such displacement non-linear shell equations is that their solution gives us the complete solution of the problem in terms of well defined and easily interpretable fields. When displacements in $\mathcal{U}$ and on $\mathcal{C}$ are determined from the shell equations, other field variables such as strain measures, rotations, stress measures etc. are calculated by the prescribed algebraic and differential procedures.

4.1. Lagrangian displacement shell equations

Since displacements and their surface derivatives appear explicity in the definitions (3.21), the set (3.23) and (3.24) of the Lagrangian shell equations can only be solved in terms of displacements as basic independent variables. Component form of $(3.23)_1$ in the undeformed basis a_α, n is given by

$$T^{\lambda\beta}\big|_\beta - b^\lambda_\beta T^\beta + p^\lambda + g^\lambda = 0 \quad ,$$

$$T^\beta\big|_\beta + b_{\lambda\beta}T^{\lambda\beta} + p + g = 0 \quad ,$$

(4.1)

where

$$g^\lambda = (n^\lambda B^\beta)\big|_\beta - b^\lambda_\beta n B^\beta \quad , \quad g = (nB^\beta)\big|_\beta + b^\lambda_\beta n_\lambda B^\beta \quad ,$$

$$B^\beta = (h_\alpha 1^\alpha_{.\,\varkappa} + h\varphi_\varkappa)\bar{a}^{\varkappa\beta} \quad ,$$

(4.2)

and the relations (3.18) or (3.21) should be introduced.

The Lagrangian equilibrium equations (4.1) and the corresponding static boundary conditions $(3.23)_{2,3}$ are linear in $N^{\alpha\beta}$, $M^{\alpha\beta}$ but are non-linear non-rational expressions in terms of displacements and their surface derivatives. When the constitutive equations (3.34) together with the strain-displacement relations $(2.7)_2$, (2.9) are introduced into (4.1) we obtain three extremely complex non-linear equations which are non-rational in terms of displacements and

their surface derivatives. These complex displacement shell equations are two-dimensionally exact for the shell reference surface.

Within the geometrically non-linear theory of shells, when strains are omitted with respect to the unity, we have

$$d \approx 1 + \gamma_\alpha^\alpha \approx 1 , \quad n \approx m (1 - \gamma_\alpha^\alpha) \approx m , \quad n_\mu \approx m_\mu ,$$

$$\varkappa_{\alpha\beta} \approx l_{\lambda\alpha}(m^\lambda|_\beta - b_\beta^\lambda m) + \varphi_\alpha(m,_\beta + b_\beta^\lambda m_\lambda) + b_{\alpha\beta}(1 + \gamma_\varkappa^\varkappa) , \qquad (4.2)$$

$$\bar{n} \approx \frac{1}{1-c_\nu^2} [n_\nu \bar{a}_t \times (\nu \times \bar{a}_t) + \sqrt{1-n_\nu^2-c_\nu^2}\, \nu \times \bar{a}_t] .$$

If $(4.2)_2$ is used in the left-hand side of (3.12) it generates the following reduced definitions of (3.21), [201,197], and of **g**

$$T^{\lambda\beta} = l_{\bullet\alpha}^\lambda(N^{\alpha\beta} + a^{\alpha\beta}b_{\varkappa\rho}M^{\varkappa\rho}) + (m^\lambda|_\alpha - b_\alpha^\lambda m)M^{\alpha\beta} + \varepsilon^{\alpha\beta}\varepsilon^{\lambda\mu}(Al_{\mu\alpha} - A_\mu\varphi_\alpha) ,$$

$$T^\beta = \varphi_\alpha(N^{\alpha\beta} + a^{\alpha\beta}b_{\varkappa\rho}M^{\varkappa\rho}) + (m,_\alpha + b_\alpha^\lambda m_\lambda)M^{\alpha\beta} + \varepsilon^{\alpha\beta}\varepsilon^{\lambda\mu}A_\mu l_{\lambda\alpha} , \qquad (4.3)$$

$$\mathbf{g} = \{(ha^{\lambda\beta}l_{\bullet\alpha}^\alpha)a_\lambda + [h_\alpha(l^{\alpha\beta} - a^{\alpha\beta}l_{\bullet\lambda}^\lambda) + h\varphi^\beta]n\}|_\beta = g^\lambda a_\lambda .$$

Therefore, in the geometrically non-linear theory of shells the Lagrangian equilibrium equations (4.1) with (4.3) are linear in $M^{\alpha\beta}$, $N^{\alpha\beta}$ and quadratic in u_α,w and their surface derivatives, while the Lagrangian static boundary conditions $(3.23)_{2,3}$ with (3.17), (4.3) and $(4.2)_3$ are linear in $N^{\alpha\beta}$, $M^{\alpha\beta}$ but still non-rational in u, n_ν, since in the reduced expression for $\bar{n}$ in $(4.2)_3$ there still remains the square-root function of the displacement parameters.

It is interesting to note that when the reduced expression $(4.2)_3$ for $\bar{n}$ is used in the right-hand side of (3.12) then analytically derived expressions for the generalized static boundary resultants **P**, **M**, $\mathbf{F}_j$ will not exactly coincide with the ones which could be constructed by omitting in (3.17) some terms which are small with respect to the unity. However, this discrepancy lies within the error margin of the first-approximation theory described by the error of the strain energy density $(3.32)_1$. As a result, both ways of deriving the reduced

Lagrangian shell equations in terms of displacements should be regarded as equivalent within the first-approximation geometrically non-linear theory of shells.

4.2. Variational principles

In many cases of practical importance it is more convenient to formulate the Lagrangian non-linear theory of shells in the variational form, as the problem of stationarity of some functional, which may be free or subjected to additional subsidiary conditions. Stationarity conditions of such a functional are then equivalent to some set of basic shell equations.

The possibility of the construction of such a functional depends upon the type of external surface and boundary loads. In general, the vector fields $\mathbf{p}$, $\mathbf{h}$, $\mathbf{T}$ and $\mathbf{H}$ may be assumed to depend arbitrarily upon the shell deformation. Such loads may be non-conservative, in general, i.e. they may not be derivable as gradients of some potentials. However, in several special cases of practical importance the external loads can be given in terms of the scalar fields $\Phi[\mathbf{u},\boldsymbol{\beta}(\nabla\mathbf{u})]$ and $\Psi[\mathbf{u},\boldsymbol{\beta}(\mathbf{u},\mathbf{u}',n_\nu)]$ by

$$\mathbf{p} = - \frac{\partial\Phi}{\partial\mathbf{u}} \quad , \quad \mathbf{h} = - \frac{\partial\Phi}{\partial\boldsymbol{\beta}} \quad , \quad \mathbf{T} = - \frac{\partial\Psi}{\partial\mathbf{u}} \quad , \quad \mathbf{H} = - \frac{\partial\Psi}{\partial\boldsymbol{\beta}} \quad . \tag{4.4}$$

When all the external loads do not depend upon the shell deformation, i.e. they are dead, they can be derived using (4.4) from the following simple potentials [201,197]

$$\Phi = - \mathbf{p}\cdot\mathbf{u} - \mathbf{h}\cdot\boldsymbol{\beta} \quad , \quad \Psi = - \mathbf{T}\cdot\mathbf{u} - \mathbf{H}\cdot\boldsymbol{\beta} \quad . \tag{4.5}$$

In case of uniformly distributed surface load of the pressure-type we may set $\bar{\mathbf{p}}(\mathbf{u}) = \bar{p}\bar{\mathbf{n}}$, where $\bar{p} = \text{const}$, but measured per unit area of $\mathcal{M}$. Then the existence of a potential depends upon the type of geometric boundary conditions. When the shell is closed [116] or when two of three displacement components are prescibed on $\mathcal{C}_u$ [168,302] then the pressure load is derivable according to $(4.4)_1$ from the potential

$$\Phi = - \bar{p}(\mathbf{n} + \frac{1}{2}\epsilon^{\alpha\beta}\mathbf{a}_\alpha \times \mathbf{u},_\beta + \frac{1}{6}\epsilon^{\alpha\beta}\mathbf{u},_\alpha \times \mathbf{u},_\beta)\cdot\mathbf{u} \quad . \tag{4.6}$$

Potentiality of different displacement-dependent surface loads is discussed in [38,226,227,231,210]. Potentiality conditions for the boundary couple $K = \bar{n} \times H$ are discussed in [9,244,269,271]. General problems associated with potential loads, treated as non-linear operators acting from the spaces of geometric variables to the conjugate force spaces, are discussed in [285,231,210].

If the external loads are derivable from potentials then the principle of virtual displacements (3.12) can be transformed into the variational principle $\delta I = 0$ for the functional

$$I = \iint_{\mathcal{M}} \{\Sigma(\gamma,\varkappa) + \Phi[u,\beta(\nabla u)]\}dA + \int_{\mathcal{C}_f} \Psi[u,\beta(u,u',n_\nu)]ds , \qquad (4.7)$$

where the strain-displacement relations $(2.7)_2$ and (2.9) as well as the geometric boundary conditions (3.24) have to be imposed as subsidiary conditions. The variational principle $\delta I = 0$ states that among all possible values of displacement and strain fields, which are subjected to the subsidiary conditions, the actual solution renders the functional (4.7) stationary.

Let us introduce the subsidiary conditions $(2.7)_2$, (2.9) and (3.24) into the functional (4.7) by using the method of Lagrange multipliers. Then we obtain the free functional

$$I_1 = \iint_{\mathcal{M}} \{\Sigma(\gamma,\varkappa) + \Phi[u,\beta(u)] - N\cdot[\gamma - \gamma(u)] - M\cdot[\varkappa - \varkappa(u)]\}dA + \qquad (4.8)$$

$$+ \int_{\mathcal{C}_f} \Psi[u,\beta(u,n_\nu)]ds - \int_{\mathcal{C}_u} [P\cdot(u-u^*) + M(n_\nu-n_\nu^*)]ds - \sum_i F_i\cdot(u_i - u_i^*) .$$

The functional I_1 is defined on three types of independent fields: displacement measures u , strain measures ε and Lagrangian multipliers σ (stress measures) defined by

$$u \equiv \{u \text{ in } \mathcal{M} ; u,n_\nu \text{ on } \mathcal{C} ; u_i \text{ at each } M_i\} ,$$

$$\epsilon \equiv \{\gamma, \varkappa \text{ in } \mathscr{U}\} ,$$ (4.9)

$$\sigma \equiv \{N, M \text{ in } \mathscr{U} ; P, M \text{ on } \mathscr{C}_u; F_i \text{ at each } M_i\} .$$

The associated Hu-Washizu (within the non-linear elasticity, for dead body and surface forces, the principle was given by Teregulov [273], extending the principles of Hu [93] and Washizu [288] of the linear elasticity) variational principle $\delta I_1 = 0$ states that among all possible values of displacement, strain and stress fields (u, ϵ, σ) which are not restricted by any subsidiary conditions, the actual solution renders the functional (4.8) stationary. Stationarity conditions of I_1 are: equilibrium equations (4.1), strain-displacement relations $(2.7)_2$ and (2.9), static boundary and corner conditions $(3.23)_{2,3}$, geometric boundary and corner conditions (3.24) and additional relations which identify the Lagrange multipliers with the fields already described by their symbolds in (4.8). These additional relations are constitutive equations (3.34), definitions of the effective generalized boundary force and couple resultants $(3.17)_{1,2}$ and definitions of the effective corner forces $(3.17)_4$.

The free three-field functional I_1 was originally constructed by PIETRASZKIEWICZ and SZWABOWICZ [201,202] using the modified tensor of change of curvature $\chi_{\alpha\beta}$ given by $(3.39)_3$ and for dead-load type external surface and boundary loads. It was also given in [218] using $K_{\alpha\beta}$ defined by $(3.39)_2'$, in [197,198,221,223] using $\varkappa_{\alpha\beta}$ and in [97] using $\hat{\rho}_{\alpha\beta}$ defined by $(3.39)_1$. Each of those formulations of I_1, which are equivalent within the first-approximation theory, can be used as a starting point for derivation of various free or constrained variational functionals, according to the general procedure discussed in [48,179,289,2,276,277]. Various functionals defined on different three and two fields as well as functionals defined on the displacement field alone were constructed by SZWABOWICZ [267,268] and SCHMIDT [217-219] for dead-load type external surface and boundary loads and by SZWABOWICZ [271] for conservative p, T and $K = \bar{n} \times H$.

Several variational functionals were also constructed by GALIMOV [67,71,72] and MUSHTARI and GALIMOV [157] in terms of the formally defined geometric boundary parameter Ω such that $\delta\Omega = \bar{v}\cdot\delta\bar{n}$. We shall prove in the section 4.4 that such a parameter does not exist, since the kinematic constraint $\bar{v}\cdot\delta\bar{n} = 0$ is not integrable, in general. As a result, the functionals given in

[67,71,72,157] in terms of Ω are meaningless within the general geometrically non-linear theory of thin elastic shells expressed in terms of displacements as basic independent variables.

4.3. Consistent classification of displacement equations for shells undergoing restricted rotations

The set of Lagrangian non-linear shell equations expressed in displacements given in section 4.1 is extremely complex even in tensor notation. This is caused by generality of those relations, since no restrictions have been imposed on displacements and/or rotations of the shell material elements. In many engineering problems of the flexible shells displacements and/or rotations can not be arbitrary, due to implicit constrains imposed by the shell geometry, limits of an elastic behaviour of the material, types of external loadings, boundary conditions etc.

Several approximation schemes leading to simplified sets of displacement shell equations were proposed in the literature. In [157,71,128,215,115,186,187] restrictions of components of the linearized rotation vector and of the displacement gradients were used to derive several simplified versions of the non-linear shell equations. Among the best known simplified versions obtained in this way are displacement shell equations of medium bending given by MUSHTARI and GALIMOV [157], for moderately small rotations proposed by SANDERS [215] and with small finite deflections derived by KOITER [115], the special case of which are the non-linear equations of shallow shells developed earlier in [144,148,295]. A variety of simplified versions proposed by DUSZEK [56,57] followed from restrictions of displacements and their surface derivatives, while those given by NOVOTNY [171] were obtained from three-dimensional equations by a formal asymptotic procedure.

The deformation about a point of the shell middle surface can be exactly decomposed into a rigid-body translation, a pure stretch along principal directions of strain and a rigid-body rotation [5,247,184,185]. Within the first-approximation theory discussed in this report strains are already assumed to be small, what leads to reduced shell relations $(3.32)_1$, (3.34), (4.1) – (4.3). Therefore, several consistently approximated versions of the non-linear displacement shell equations were constructed in [185,190] by imposing additional restrictions upon the finite rotations of the shell material elements.

A finite rotation in the shell may be described by the angle of rotation ω about an axis of rotation described by the unit vector e. The rotations in [185,190] were classified in terms of the small parameter θ defined in (3.33) as follows: a) $\omega \leq O(\theta^2)$ - small rotations, $\omega = O(\theta)$ - moderate rotations (cf. [207]), $\omega = O(\sqrt{\theta})$ - large rotations, $\omega \geq O(1)$ - finite rotations. This classification restricts the magnitude of the rotation angle ω. However, shell structures are usually quite rigid for in-surface deformation being flexible for out-of-surface deformation. In order to take this into account the finite rotation vector $\Omega = e\sin\omega$ may be defined. Since for $|\omega| < \pi/2$, $O(|\Omega|) = O(\sin\omega) = O(\omega)$ the name "small, moderate, large or finite rotation" may be associated with the particular component $\Omega = \Omega \cdot n$ or $\Omega_\beta = \Omega \cdot a_\beta$ of Ω.

Within small strains (but not small rotations) the vector Ω is expressed in terms of displacements by [185,193]

$$\Omega \approx \varepsilon^{\beta\alpha}[\varphi_\alpha(1 + \frac{1}{2}\theta^\varkappa_\varkappa) - \frac{1}{2}\varphi^\lambda(\theta_{\lambda\alpha} - \omega_{\lambda\alpha})]a_\beta + \frac{1}{2}\varepsilon^{\alpha\beta}\omega_{\alpha\beta}n \ . \tag{4.10}$$

For any restriction imposed on Ω estimates for φ_α and $\omega_{\alpha\beta}$ are given by (4.10) and estimates for $\theta_{\alpha\beta}$ follow from $(2.7)_2$ with $\gamma_{\alpha\beta} = O(\eta)$. Then simplified expressions for the strain measures $\gamma_{\alpha\beta}$ and $\varkappa_{\alpha\beta}$ can be obtained taking into account the accuracy of the strain energy density $(3.32)_1$. In the estimation procedure covariant surface derivatives are estimated by dividing their maximal value by a large parameter λ defined by

$$\lambda = \frac{h}{\theta} = \min(b, L, 1, \sqrt{hR}, \frac{1}{\sqrt{\eta}}) \ . \tag{4.11}$$

Introducing such energetically consistent simplified expressions of the strain measures into the Lagrangian principle of virtual displacements (3.12) one gets the corresponding reduced expressions for the internal force vector T^β and the generalized static boundary parameters P, M and F_j, together with consistently simplified expression for the geometric boundary parameter n_ν.

Simplified versions of the Lagrangian shell equations proposed in [183] were discussed in [185,193]. Simplifications of the entirely Lagrangian shell equations derived in [201] were given in detail in [195,197]. Let us remind here some of those consistently approximated non-linear shell equations.

Within **small rotations** $\varphi_\alpha = O(\theta^2)$, $\omega_{\alpha\beta} = O(\theta^2)$, $\theta_{\alpha\beta} = O(\theta^2)$ and the strain

measures are approximated by $\gamma_{\alpha\beta} = \vartheta_{\alpha\beta} + O(\eta\theta^2)$, $\varkappa_{\alpha\beta} = -\frac{1}{2}(\varphi_{\alpha|\beta} + \varphi_{\beta|\alpha}) + O(\eta\theta/\lambda)$, which describe the linear bending theory of shells treated extensively in many monographs.

Within **moderate rotations** $\varphi_\alpha = O(\theta)$, $\omega_{\alpha\beta} = O(\theta)$, $\vartheta_{\alpha\beta} = O(\theta^2)$ and the consistently simplified shell relations take the form [185,193,224]

$$\gamma_{\alpha\beta} = \vartheta_{\alpha\beta} + \frac{1}{2}\varphi_\alpha\varphi_\beta + \underline{\frac{1}{2}\omega^\lambda{}_{\cdot\alpha}\omega_{\lambda\beta} - \frac{1}{2}(\vartheta^\lambda_\alpha\vartheta_{\lambda\beta} + \vartheta^\lambda_\beta\omega_{\lambda\alpha})} + O(\eta\theta^2) \ ,$$

(4.12)

$$\varkappa_{\alpha\beta} = -\frac{1}{2}(\varphi_{\alpha|\beta} + \varphi_{\beta|\alpha} \underline{- b^\lambda_\alpha\omega_{\lambda\beta} - b^\lambda_\beta\omega_{\lambda\alpha}}) + O(\frac{\eta\theta}{\lambda}) \ ,$$

$$\mathbf{T}^\beta = [N^{\lambda\beta} - \frac{1}{2}(b^\lambda_\alpha M^{\alpha\beta} + b^\beta_\alpha M^{\alpha\lambda}) \underline{- \frac{1}{2}\omega^{\lambda\beta}N^\alpha_\alpha - \frac{1}{2}(\omega^{\lambda\alpha}N^\beta_\alpha + \omega^{\beta\alpha}N^\lambda_\alpha)} +$$

$$+ \underline{\frac{1}{2}(\vartheta^{\lambda\alpha}N^\beta_\alpha - \vartheta^{\beta\alpha}N^\lambda_\alpha)}]a_\lambda + (\varphi_\alpha N^{\alpha\beta} + M^{\alpha\beta}|_\alpha)n + h^\beta n \ ,$$

(4.13)

$$\bar{\mathbf{n}} = -\varphi_\alpha a^\alpha + n + O(\theta^2) \ , \quad n_\nu = -\varphi_\nu \ ,$$

$$\mathbf{F} = M_{t\nu}n \ , \quad M = M_{\nu\nu} \ .$$

If, additionally, rotations about the normal are assumed to be also small then also $\omega_{\alpha\beta} = O(\theta^2)$. For such **moderate/small rotation** theory of shells the relations (4.12) and (4.13) may be considerably simplified by omitting there the underlined terms.

The set of non-linear relations (3.23), (3.24) with (4.13) and (4.12) describes the consistently reduced Lagrangian non-linear theory of shells under-going moderate rotations. The theory contains as special cases various simpler versions of shell equations proposed in the literature. Among them are the theory of medium bending [157], for moderately small rotations [215], with small finite deflections [115] and the classical non-linear theory of shallow shells. A detailed review of those simpler versions was presented by SCHMIDT and PIETRASZKIEWICZ [224], where also a set of sixteen basic free functionals and several functionals with subsidiary conditions was constructed for conservative dead-type surface and boundary loadings, (cf. [216]). These functionals and associated with them variational principles extend to the moderate rotation range of deformation earlier results on particular variational principles formulated

for shallow shells [296,6,157,287,94,74,255,3,256,67,34,249,71] and for simplified versions of the theory of shells undergoing moderate rotations [258,259,251]. Stability equations for the moderate rotation theory of shells are given in [139,260], which extend various simpler versions of stability equations given in the literature. More complex moderate-rotation shell equations were proposed in [23,26,163], where the expression for $\varkappa_{\alpha\beta}$ contains also some non-linear terms, whose contribution to the strain energy density $(3.32)_1$ lies within the indicated error of the first-approximation theory.

Within **large rotations** $\varphi_\alpha = O(\sqrt{\theta})$, $\omega_{\alpha\beta} = O(\sqrt{\theta})$, $\theta_{\alpha\beta} = O(\theta)$. Appropriately simplified relations of the Lagrangian theory of [183] were discussed already in [185,193]. It was found in [193] that the consistently simplified (but still non-linear) expression for $\varkappa_{\alpha\beta}$ generated the boundary integral, which contained six (instead of four) independent variations: δu and $\delta(\frac{du}{ds})_v$. This did not allow for a variational formulation of the shell problem even if the external boundary forces were conservative. An explanation for this paradox was found in the definition of the fourth geometric boundary parameter used in [183,185], which was not entirely Lagrangian. As a result, an entirely Lagrangian non-linear theory of shells was proposed in [201] where the new parameter n_v was used on the shell boundary. Appropriately simplified relations of [201] within the large rotation range of deformation were discussed in detail in [195], various alternative results, within the prescribed accuracy of the strain energy density, were presented also in [219-222,197,198,165,169,170,140,141].

The most interesting special case of the large rotation shell theory appears when rotations about the normal are assumed to be always small, i.e. $\omega_{\alpha\beta} = O(\theta^2)$. If, additionally, we allow for a greater error in the strain energy function $(3.32)_1$ to be $O(Eh\eta^2\theta\sqrt{\theta})$ instead of $O(Eh\eta^2\theta^2)$ then the set of shell equations for such **simplified large/small rotation** theory (without **h**) is described by the following relations [197,198]

$$\gamma_{\alpha\beta} = \theta_{\alpha\beta} + \frac{1}{2}\varphi_\alpha\varphi_\beta + \frac{1}{2}\theta^\lambda_\alpha\theta_{\lambda\beta} - \frac{1}{2}(\theta^\lambda_\alpha\omega_{\lambda\beta} + \theta^\lambda_\beta\omega_{\lambda\alpha}) + O(\eta\theta\sqrt{\theta}) \ ,$$

$$\varkappa_{\alpha\beta} = -\frac{1}{2}\{[(\delta^\lambda_\alpha + \theta^\lambda_\alpha)\varphi_{\lambda|\beta} + (\delta^\lambda_\beta + \theta^\lambda_\beta)\varphi_{\lambda|\alpha}] - \varphi^\lambda(\varphi_\alpha\varphi_{\lambda|\beta} + \varphi_\beta\varphi_{\lambda|\alpha}) +$$

$$+ (b^\lambda_\alpha\theta_{\lambda\beta} + b^\lambda_\beta\theta_{\lambda\alpha}) + (b^\lambda_\alpha\varphi_\beta + b^\lambda_\beta\varphi_\alpha)\varphi_\lambda - b_{\alpha\beta}\varphi^\lambda\varphi_\lambda\} + O(\frac{\eta\sqrt{\theta}}{\lambda}) \ ,$$

$$(4.14)$$

$$n_\nu = -\varphi_\nu + O(\theta^2\sqrt{\theta}), \quad \bar{n} \approx -\varphi_\nu\nu - \varphi_t t + (1 - \frac{1}{2}\varphi_\nu^2 - \frac{1}{2}\varphi_t^2)n ,$$

$$T^{\lambda\beta} = (\delta_\alpha^\lambda + \vartheta_\alpha^\lambda)N^{\alpha\beta} - \frac{1}{2}(\underline{\omega^{\lambda\alpha}N_\alpha^\beta + \omega^{\beta\alpha}N_\alpha^\lambda}) - \frac{1}{2}[(b_\alpha^\lambda + \varphi^\lambda|_\alpha)M^{\alpha\beta} + \underline{(b_\alpha^\beta + \varphi^\beta|_\alpha)M^{\alpha\lambda}}] ,$$

$$T^\beta = \varphi_\alpha N^{\alpha\beta} + [(\delta_\lambda^\beta + \vartheta_\lambda^\beta)M^{\lambda\alpha}]|_\alpha + (\varphi_\lambda M^{\lambda\alpha})|_\alpha\varphi^\beta - \varphi^\lambda|_\alpha\varphi_\lambda M^{\alpha\beta} -$$

$$- \underline{(b_\alpha^\lambda M^{\alpha\beta} + b_\alpha^\beta M^{\alpha\lambda})\varphi_\lambda + b_{\alpha\lambda}\varphi^\beta M^{\alpha\lambda}} ,$$

<div align="right">(4.15)</div>

$$R_{\nu\nu} = (1 + \vartheta_{\nu\nu})M_{\nu\nu} + \vartheta_{\nu t}M_{t\nu} , \quad R_{t\nu} = \vartheta_{t\nu}M_{\nu\nu} + (1 + \vartheta_{tt})M_{t\nu} ,$$

$$R_\nu = \varphi_\nu M_{\nu\nu} + \varphi_t M_{t\nu} , \quad M = (1 + \vartheta_{\nu\nu} + \varphi_\nu^2)M_{\nu\nu} + (\vartheta_{\nu t} + \varphi_\nu\varphi_t)M_{t\nu} ,$$

$$\mathbf{F} = F\mathbf{n} , \quad F = (\vartheta_{\nu t} + \varphi_\nu\varphi_t)M_{\nu\nu} + (1 + \vartheta_{tt} + \varphi_t^2)M_{t\nu} .$$

The relations (4.14) and (4.15) have the important property: for conservative surface and boundary loadings they allow to contruct the functional (4.8), which stationary conditions lead **exactly** to all shell equations described by (4.14) and (4.15). Another such formulation was proposed in [197,221]. Alternative versions of shell relations of the simplified large/small rotation theory discussed in [195,220,165,170] are also energetically consistent, although some additional transformations should be applied in order to derive the shell equations from the variational functional (4.8). In particular, the version proposed by NOLTE and STUMPF [170], in which $\varkappa_{\alpha\beta}$ are quadratic polynomials in displacements and its surface derivatives, was shown [165,169,140,141,166,167] to be numerically efficient and leading to good results also far beyond the large rotation range of shell deformation.

In some engineering applications the shell relations (4.14) and (4.15) may still be simplified at the expence of a larger loss in accuracy of (3.32)$_1$ to be $O(Eh\eta^2\theta)$. Within this larger error the shell relations of such **simplest large/small rotation** theory of shells [197] are described again by (4.14) and (4.15), where underlined terms should be omitted and the term $\vartheta_\alpha^\lambda N^{\alpha\beta}$ in (4.15)$_1$ should be replaced by its symmetric part $\frac{1}{2}(\vartheta_\alpha^\lambda N^{\alpha\beta} + \vartheta_\alpha^\beta N^{\alpha\lambda})$. Alternative energetically consistent versions were proposed in [193,195,196,165,170,218,221]. On the other hand, the comparative discussion given in [198] suggests that some

known versions [115,23,71,235] of the non-linear theory of shells, which are based on various quadratic expressions of $\varkappa_{\alpha\beta}$, can not be regarded as energetically consistent within the large-rotation range of deformation, since some energetically important terms $O(\theta\sqrt{\theta}/\lambda)$ do not appear in the expressions for $\varkappa_{\alpha\beta}$ used there. Verious simplified versions of the non-linear shell relations were also proposed in [99,100,58,87,88,98,45,46,230,31].

When only rotations about the normal are assumed to be small, while other ones are unrestricted, then $\varphi_\alpha = O(1)$, $\omega_{\alpha\beta} = O(\theta^2)$, $\vartheta_{\alpha\beta} = O(1)$. For such **finite/small rotation** theory of shells only few terms may be omitted in $\varkappa_{\alpha\beta}$ within the error $O(Eh\eta^2\theta^2)$ of $(3.32)_1$ or even within the greater error $O(Eh\eta^2\theta\sqrt{\theta})$. It seems, therefore, that considerably simplified shell relations derived in [104,236] for such a theory cannot be justified within the assumed error of the first-approximation theory.

An extensive comparative numerical analysis, based on energetically consistent simplified versions of non-linear shell equations discussed above and on several other simplified versions proposed in the literature, was carried out in the series of papers [139-141,252,253,165-169,35,59,89,90,86,47] for a large number of one- and two-dimensional problems of flexible shells. In order to provide a reliable reference solution the full version of entirely Lagrangian shell equations [201] and in [47] also the refined three-dimensional NONSAP numerical code were used. The results of the numerical analysis showed that all energetically consistent versions of non-linear shell equations led to results which, within the range of their applicability, were always in a good agreement with the reference solution. In some examples the agreement was adequate also far beyond the range of applicability of those versions. On the other hand, some of the simplified versions suggested in the literature, which were even more complex but still energetically inconsistent, led to load-displacement paths which occasionally diverged from the reference path already on an early stage of the shell deformation.

4.4. Integrability of kinematic boundary constraints

In the entirely Lagrangian non-linear theory of shells discussed in section 3.2 the component $n_\nu = \bar{n}\cdot\nu$ has been used as the fourth independent boundary parameter, in terms of which $\delta\bar{n}$ can be given [201,197] by

$$\delta \bar{n} = \frac{1}{a_\nu}[\bar{a}_\nu \delta n_\nu + \mathbf{v} \times \bar{n}(\bar{n} \cdot \delta \bar{r}')] \ . \tag{4.16}$$

This has allowed to reduce (3.12) into (3.15) and to construct four work-conjugate static $(3.23)_{2,3}$ and geometric (3.24) boundary conditions.

In the derivation of the mixed shell equations in section 3.3 an alternative expression for $\delta \bar{n}$ has been used

$$\delta \bar{n} = \bar{v}(\bar{v} \cdot \delta \bar{n}) - \frac{1}{\bar{a}_t} \bar{r}'(\bar{n} \cdot \delta \bar{r}') \ . \tag{4.17}$$

This has allowed to reduce (3.27) into (3.28).

Still another expression for $\delta \bar{n}$ results from a direct variation of $(2.18)_1$ to be

$$\delta \bar{n} = - \nu_\beta \bar{a}^\beta (\bar{n} \cdot \delta \bar{r}_{,\nu}) - t_\beta \bar{a}^\beta (\bar{n} \cdot \delta \bar{r}') \ , \tag{4.18}$$

$$\nu_\beta \bar{a}^\beta = \bar{a}_t \frac{1}{\bar{d}} \bar{v} \ , \qquad t_\beta \bar{a}^\beta = \frac{1}{\bar{a}_t} (\bar{t} - \frac{1}{\bar{d}} 2\gamma_{\nu t} \bar{v}) \ .$$

When (4.18) is introduced into $(3.12)_2$ the internal boundary integral transforms into

$$\int_{\mathcal{C}} \{[T^\beta \nu_\beta + (M_{\nu t} \bar{n})'] \cdot \delta \bar{r} - M_{\nu\nu} \bar{n} \cdot \delta \bar{r}_{,\nu}\} ds + \sum_j (M_{\nu t} \bar{n})_j \cdot \delta \bar{r}_j = \tag{4.19}$$

$$= \int_{\mathcal{C}} \{[T^\beta \nu_\beta + M_{\nu\nu} \bar{n}_{,\nu} + (M_{\nu t} \bar{n})'] \cdot \delta \bar{r} - M_{\nu\nu} (\bar{n} \cdot \delta \bar{r})_{,\nu}\} ds + \sum_j (M_{\nu t} \bar{n})_j \cdot \delta \bar{r}_j \ .$$

The transformed line integral $(4.19)_2$ was used in [183,304] while the simpler integral $(4.19)_1$ was not used in the literature.

Static boundary and corner conditions on $\mathcal{C}_f$ have been constructed in (3.23) and (3.30) by demanding that all the multipliers of $\delta \bar{r}$, $\delta \bar{r}_j$ and of δn_ν or $\bar{v} \cdot \delta \bar{n}$ in the line integral should identically vanish. Using the transformation

(4.18) and (4.19)$_1$ we may construct alternative static boundary and corner conditions on $\mathcal{C}_f$ again by demanding that all the multipliers of $\delta\bar{r}$, $\delta\bar{r}_j$ and $\bar{n}\cdot\delta\bar{r}_{,\nu}$ should identically vanish. It is implicitly assumed that the work-conjugate geometric boundary conditions on $\mathcal{C}_u$ should satisfy the kinematic constraints $\delta\bar{r} = 0$, $\delta\bar{r}_i = 0$ and $\delta n_\nu = 0$, $\bar{v}\cdot\delta\bar{n} = 0$ or $\bar{n}\cdot\delta\bar{r}_{,\nu} = 0$,respectively. It is easy to note that from the kinematic constraints $\delta\bar{r} = 0$, $\delta n_\nu = 0$ and $\delta\bar{r}_i = 0$ follow the geometric boundary conditions $\bar{r} = \bar{r}^*$, $n_\nu = n_\nu^*$ on $\mathcal{C}_u$ and $\bar{r}_i = \bar{r}_i^*$ on each $M_i \in \mathcal{C}_u$. It is not apparent, however, what kind of a scalar parameter should be assumed to be given on $\mathcal{C}_u$ in order to satisfy the fourth kinematic constraints $\bar{v}\cdot\delta\bar{n} = 0$ or $\bar{n}\cdot\delta\bar{r}_{,\nu} = 0$. Therefore, the question arises whether there exists a scalar parameter ϕ such that its variation on $\mathcal{C}_f$ would coincide with the variational expressions $\bar{v}\cdot\delta\bar{n}$ or $\bar{n}\cdot\delta\bar{r}_{,\nu}$, possibly multiplied by some scalar function μ . If such functions ϕ and μ exist, the question arises how to construct them. This general problem has been solved only recently by MAKOWSKI and PIETRASZKIEWICZ [142]. Here we summarize some of the results given there.

The variational expressions $\bar{v}\cdot\delta\bar{n}$, $\bar{n}\cdot\delta\bar{r}_{,\nu}$ or δn_ν discussed above are particular cases of the following general variational expression

$$\omega = A\cdot\delta\bar{r}_{,\nu} + B\cdot\delta\bar{r}' \ , \tag{4.20}$$

where $A = A(\bar{r}_{,\nu},\bar{r}')$ and $B = B(\bar{r}_{,\nu},\bar{r}')$ are vector-valued functions of the vector arguments.

Extending the method suggested in [304], it was shown in [142] that at each point $M \in \mathcal{C}$ the variational expression (4.20) may be regarded as a differential one-form on the six-dimensional manifold X with local coordinates $\xi_i \in X$, $i=1,2,...,6$ defined by

$$\xi_i = (v\cdot\bar{r}_{,\nu} \ , \ t\cdot\bar{r}_{,\nu} \ , \ n\cdot\bar{r}_{,\nu} \ ,v\cdot\bar{r}' \ , \ t\cdot\bar{r}' \ , \ n\cdot\bar{r}') \ . \tag{4.21}$$

Let also components of (A,B) in the basis v,t,n be defined by

$$A_i = (v\cdot A \ , \ t\cdot A \ , \ n\cdot A \ , \ v\cdot B \ , \ t\cdot B \ , \ n\cdot B) \ , \tag{4.22}$$

so that $\omega = \sum\limits_{i=1}^{6} A_i \delta\xi_i$.

The one-form (4.20) is said to be exact if there exists a primitive scalar-valued function $\phi(\bar{r}_{,\nu},\bar{r}')$ such that $\omega = \delta\phi$. The necessary conditions for ω to be exact are

$$A_{j,i} - A_{i,j} = 0 \qquad (4.23)$$

for any $i,j \in (1,2,\ldots,6)$. The one-form (4.20) is said to be integrable if there exist scalar-valued functions $\mu(\bar{r}_{,\nu},\bar{r}')$, called the integrating factor, and $\phi(\bar{r}_{,\nu},\bar{r}')$ such that $\mu\omega = \delta\phi$. The necessary conditions for ω to be integrable are

$$A_i(A_{k,j} - A_{j,k}) + A_j(A_{i,k} - A_{k,i}) + A_k(A_{j,i} - A_{i,j}) = 0 \qquad (4.24)$$

for any $i,j,k \in (1,2,\ldots,6)$.

Let us check the exactness and integrability of the one-form $\omega = \bar{v} \cdot \delta\bar{n}$, for which A and B are given by

$$A = - \bar{a}_t \frac{1}{d} \bar{n} , \qquad B = \frac{1}{\bar{a}_t} \frac{1}{d} 2\gamma_{\nu t}\bar{n} . \qquad (4.25)$$

Differentiation of (4.25) with respect to $\bar{r}_{,\nu}$ and $\bar{r}'$ gives

$$\frac{\partial A}{\partial \bar{r}_{,\nu}} = \bar{a}_t^2 \frac{1}{d^2} (\bar{v} \otimes \bar{n} + \bar{n} \otimes \bar{v}) ,$$

$$\frac{\partial A}{\partial \bar{r}'} = \frac{1}{d} \bar{t} \otimes \bar{n} - \frac{1}{d^2} 2\gamma_{\nu t}(\bar{v} \otimes \bar{n} + \bar{n} \otimes \bar{v}) = [\frac{\partial B}{\partial \bar{r}_{,\nu}}]^T ,$$

$$\qquad (4.26)$$

$$\frac{\partial B}{\partial \bar{r}'} = \frac{1}{\bar{a}_t^2} \bar{n} \otimes \bar{v} + \frac{1}{\bar{a}_t^2} \frac{1}{d^2} (2\gamma_{\nu t})^2 (\bar{v} \otimes \bar{n} + \bar{n} \otimes \bar{v}) -$$

$$\qquad\qquad - \frac{1}{\bar{a}_t^2} \frac{1}{d} 2\gamma_{\nu t}(\bar{t} \otimes \bar{n} + \bar{n} \otimes \bar{t}) .$$

Since $(4.26)_3$ is not symmetric, the conditions (4.23) are not satisfied for $(i,j) = (4,5)$, for example. Moreover, with (4.25) and (4.26) the integrability

conditions (4.24) are not satisfied as well for $(i,j,k) = (1,4,5)$, for example. As a result, the differential one-form $\bar{v} \cdot \delta \bar{n}$ is neither exact nor integrable, in general. The discussion given in more detail in [142] provides the proof for the same statement given by ZUBOV [305].

The variational expression $\bar{v} \cdot \delta \bar{n}$, which has appeared originally in the paper by GALIMOV [93], may be presented in several different but equivalent forms. Note that in terms of the difference vector β given in $(2.14)_2$ $\delta \bar{n} = \delta \beta$ and $\bar{v} \cdot \delta \bar{n} = \bar{v} \cdot \delta \beta \equiv \delta \bar{\beta}_\nu$, which was used in [185,214]. Here δ should not be understood as the variation of $\bar{\beta}_\nu$, since so defined $\delta \bar{\beta}_\nu \neq \delta(\bar{v} \cdot \beta)$. The rotation of the boundary may also be described [185,188,192] by the total rotation tensor $R_t = \bar{v} \otimes v + \bar{t} \otimes t + \bar{n} \otimes n$ such that $\bar{n} = R_t n$. Then we can introduce axial vectors $\delta \omega_t$ and δw_t of the scew-symmetric tensors $\delta R_t R_t^T$ and $R_t^T \delta R_t$, respectively, according to [138,198,199],

$$\delta R_t R_t^T = \delta \omega_t \times 1 \ , \quad R_t^T \delta R_t = \delta w_t \times 1 \ , \quad \delta \omega_t = R_t \delta w_t \ . \tag{4.27}$$

Since $\delta \bar{n} = \delta \omega_t \times \bar{n} = R_t (\delta w_t \times n)$ it follows that we have $\bar{v} \cdot \delta \bar{n} = \delta \omega_t \cdot \bar{t} = \delta w_t \cdot t$. Here again δ should not be understood as the symbol of variation of ω_t or w_t , since the symbols ω_t or w_t alone have no geometric meaning here. The expression $\delta \omega_t \cdot \bar{t}$ was applied, among others, in [291,214,17] while $\delta w_t \cdot t$ was used in [271].

According to the discussion given above and in (3.6) the variational expressions $\delta \bar{\beta}_\nu$, $\delta \omega_t \cdot \bar{t}$, $\delta w_t \cdot t$, $-\bar{n} \cdot \delta \bar{v}$, $-\delta \bar{\phi}_\nu$, $-\bar{n} \cdot \delta \bar{u}_{,\bar{\nu}}$ which appeared in the literature are all equivalent to the differential one-form $\bar{v} \cdot \delta \bar{n}$, since they have all the same representation (4.20) with (4.25) in terms of variations of $\bar{r}_{,\nu}$ and $\bar{r}'$. As a result, neither of the one-forms is exact or integrable as well. It is apparent from this discussion that the variational principles given by GALIMOV [68,71] in terms of Ω such that $\delta \Omega = \bar{v} \cdot \delta \bar{n}$ are not correct, in general, since such a function Ω does not exist.

In [142] it was confirmed that the differential one-form $\omega = v \cdot \delta \bar{n} \equiv \delta n_\nu$ is exact indeed and its primitive function is $\phi \equiv n_\nu$. It was also proved that the one-form $\bar{n} \cdot \delta \bar{r}_{,\nu}$ is neither exact nor integrable, since the conditions (4.24) are not satisfied. Using the same method many other variational expressions of the type (4.20) may be checked. On the other hand, similar direct discussion of integrability of the variational expressions $(\bar{n} \cdot \delta \bar{u})_{,\bar{\nu}}$ or $(\bar{n} \cdot \delta \bar{r})_{,\nu}$ has to be

performed with the help of a nine-dimensional manifold with local coordinates identified with components of $\bar{r}$, $\bar{r}_{,\nu}$, $\bar{r}'$ in the basis ν, t, n. However, such a discussion is not necessary, since those variational expressions can always be transformed further by taking the partial derivative with respect to s_ν on $\mathcal{C}$. Since $\delta\bar{r}$ is exact on a three-dimensional manifold of positions $\bar{r}$, the problem can always be reduced to the integrability of the one-form of the type (4.20) on X.

4.5. Work-conjugate boundary conditions

Each of the variational expressions of the type (4.20), which appear in the boundary line integral and is connected with the boundary couple, may be transformed further by multiplying (and dividing) it by a non-vanishing scalar function $\eta(\bar{r}_{,\nu}, \bar{r}')$ and by adding (and subtracting) terms of the type $c(\bar{r}_{,\nu}, \bar{r}') \cdot \delta\bar{r}'$, since terms with $\delta\bar{r}'$ can always be eliminated by integration by parts. By such a transformation a non-integrable one-form may be transformed to the exact one-form, for which a primitive may be constructed.

In [142] the following simple differential one-form on the six-dimensional manifold X has been discussed

$$\theta = d \cdot \delta\bar{r}_{,\nu} \quad , \quad d = d\bar{n} = \bar{r}_{,\nu} \times \bar{r}' \quad ,$$

$$A = (\xi_2\xi_6 - \xi_3\xi_5)\nu + (\xi_3\xi_4 - \xi_1\xi_6)t + (\xi_1\xi_5 - \xi_2\xi_4)n \quad , \tag{4.28}$$

$$B = A_4\nu + A_5t + A_6n \equiv \mathbf{0} \quad .$$

It is easy to check that the one-form θ is not integrable. In [142] it has been proved that an arbitrary function $\phi(\bar{r}',\alpha)$, where $\alpha = A_1/A_3 = n_\nu/n$, is the primitive of some transformed one-form ψ such that

$$\delta\phi = \psi = \eta d \cdot \delta\bar{r}_{,\nu} + c \cdot \delta\bar{r}' \quad ,$$

$$\eta = -\frac{1}{A_3^2}\xi_5 x \quad , \quad c = \lambda + \frac{1}{A_3^2}\xi_2 x d \quad , \tag{4.29}$$

$$\lambda = \frac{\partial\phi}{\partial\bar{r}'} \quad , \quad x = \frac{\partial\phi}{\partial\alpha} \quad .$$

If we solve (4.29) for $d \cdot \delta \bar{r}_{,\nu}$ and introduce it into $(4.18)_1$ then we obtain still another general expression for $\delta \bar{n}$ to be

$$\delta \bar{n} = \nu_\alpha \bar{a}^\alpha f \delta \phi - [\nu_\beta f \lambda + (\nu_\beta g + t_\beta) \bar{n}] \bar{a}^\beta \cdot \delta \bar{r}' ,$$

(4.30)

$$f = \frac{dn^2}{c_t x} , \quad g = \frac{\xi_2}{\xi_5} = \frac{1}{\bar{a}_t^2} (\frac{1}{d} \frac{c_\nu n - c n_\nu}{c_t} + 2\gamma_{\nu t}) .$$

The expression $(4.30)_1$ is remarkable by the fact, that it is given directly in terms of variation of an arbitrary function $\phi(\bar{r}', \alpha)$. If now the expression $(4.30)_1$ for $\delta \bar{n}$ is used to transform the Lagrangian principle of virtual displacements (3.12) then it can take the form

$$- \iint_{\mathcal{M}} (T^\beta|_\beta + p) \cdot \delta \bar{r} dA + \sum_j (F_j - F_j^*) \cdot \delta \bar{r}_j +$$

(4.31)

$$+ \int_{\mathcal{C}_f} [(T^\beta \nu_\beta + F' - T - F^{*'}) \cdot \delta \bar{r}' + (M - M^*) \delta \phi] ds = 0 ,$$

where

$$F = f M_{\nu\nu} \lambda + (g M_{\nu\nu} + M_{\nu t}) \bar{n} , \quad M = f M_{\nu\nu} ,$$

(4.32)

$$F^* = (H \cdot \bar{a}^\beta)[\nu_\beta f \lambda + (\nu_\beta g + t_\beta) \bar{n}], \quad M^* = f(H \cdot \bar{a}^\beta) \nu_\beta .$$

For arbitrary $\delta \bar{r}$, $\delta \bar{r}_j$ and $\delta \phi$ from (4.31) follow the equilibrium equations $(3.23)_1$ and static boundary conditions

$$T^\beta \nu_\beta + F' = T + F^{*'} , \quad M = M^* \text{ on } \mathcal{C}_f ,$$

(4.33)

$$F_j = F_j^* \text{ at each corner } M_j \in \mathcal{C}_f .$$

Corresponding work-conjugate geometric boundary conditions are

$$\bar{r} = \bar{r}^* \; , \; \phi = \phi^* \; \text{on} \; \mathcal{C}_u \; ,$$

$$(4.34)$$

$$\bar{r}_i = \bar{r}_i^* \; \text{at each corner} \; M_i \; \varepsilon \; \mathcal{C}_u \; .$$

The arbitrariness of ϕ allows for a wide freedom in choosing the form of boundary conditions to be used in the shell theory. This enables one to choose such definition of ϕ which would suit best to a particular shell problem. In particular, it was shown in [142] that the parameters n_ν, θ_ν used in [178] and the total rotation angle ω_t of R_t are all special cases of ϕ .

5. Shell relations in terms of rotations

Some shell problems are solved in a more convenient way if one uses finite rotations together with other fields as basic independent variables of the non-linear shell equations. Already REISSNER [206,207] proposed the set of non-linear equations for an axisymmetric deformation of shells of revolution written in terms of a rotation and a stress resultant (or a stress function) as independent variables. This formulation led to a number of papers on axisymmetric problems of shells of revolution the results of which have been summarized, among others, in the books by SHILKRUT and VYRLAN [238] and LIBAI and SIMMONDS [104].

Within the general non-linear theory of thin shells ALUMÄE [5] derived the non-linear equilibrium equations and compatibility conditions in the intermediate rotated basis while SIMMONDS and DANIELSON [247,248] proposed a set of non-linear shell equations in terms of a finite rotation vector and a stress function vector as independent variables and constructed an appropriate variational principle. The theory of finite rotations in shells developed by PIETRASZKIEWICZ [184,185] led to several alternative forms of non-linear shell equations, boundary conditions, consistently approximated shell relations and some new kinematic relations which have been summarized in [186,188,190-194]. Contributions to the non-linear theory of shells in terms of rotations were given also by WEMPNER [290-293], SHAMINA [233,234], VALID [281-284], SHKUTIN [240], REISSNER [208,209], LIBAI and SIMMONDS [130], ATLURI [14], MAKOWSKI and STUMPF [143] and BADUR and PIETRASZKIEWICZ [19] where further references are given.

The primary advantage of the non-linear shell equations in terms of finite rotations is that they contain at most first derivatives of the independent field variables. In the computerized analysis of shells this allows to use the simplest shape functions or the simplest difference schemes which assure the high efficiency of the numerical analysis.

The non-linear theory of shells in terms of rotations is now in the process of development and several questions are still open. Only few two-dimensional problems have been analysed [60,61] using this approach. Therefore, we found it worthwhile to review here in more detail, in the unified notation, the most important results given in the literature and to supplement them with some new results which are not available elswhere. It is hoped that it will stimulate further research in the field.

5.1. Additional geometric relations

Applying the polar decomposition theorem [279,280,139] the deformation gradient tensor $\mathbf{G}$ defined in (2.5) can be represented [184,185,190] in the form

$$\mathbf{G} = \mathbf{RU} = \mathbf{VR} \ , \qquad \mathbf{G}^{-1} = \mathbf{U}^{-1}\mathbf{R}^{T} = \mathbf{R}^{T}\mathbf{V}^{-1} \ . \tag{5.1}$$

Here $\mathbf{U}$ and $\mathbf{V}$ are the right and left stretch tensors, respectively, while $\mathbf{R}$ is the finite rotation tensor. The tensors $\mathbf{U}$ and $\mathbf{V}$ are symmetric and positive definite while $\mathbf{R}$ is the proper orthogonal, i.e. $\det\mathbf{R} = +1$.

By (2.1) and (5.1) the deformation of a neighbourhood about a particle of the shell middle surface has been decomposed into a rigid-body translation, a pure stretch along principal directions of $\mathbf{U}$ (or $\mathbf{V}$) and a rigid-body rotation. From (2.2) and (5.1) it follows that there exist two intermediate non-holonomic bases, the stretched basis $\mathbf{s}_\alpha,\mathbf{n}$ and the rotated basis $\mathbf{r}_\alpha,\bar{\mathbf{n}}$, which are defined by

$$\mathbf{s}_\alpha = \mathbf{U}\mathbf{a}_\alpha = \mathbf{R}^{T}\bar{\mathbf{a}}_\alpha \ , \qquad \mathbf{s}_\alpha\cdot\mathbf{s}_\beta = \bar{\mathbf{a}}_{\alpha\beta} \ , \tag{5.2}$$

$$\mathbf{r}_\alpha = \mathbf{R}\mathbf{a}_\alpha = \mathbf{V}^{-1}\bar{\mathbf{a}}_\alpha \ , \qquad \mathbf{r}_\alpha\cdot\mathbf{r}_\beta = \mathbf{a}_{\alpha\beta} \ . \tag{5.3}$$

Within the shell theory the rotated basis $\mathbf{r}_\alpha,\bar{\mathbf{n}}$ was introduced first by ALUMÄE [5] and was used in [8,247,248,240,133,19]. The stretched basis $\mathbf{s}_\alpha,\mathbf{n}$ was introduced first by NOVOZHILOV and SHAMINA [178] and used in [184,185,188,190-194,14]. In terms of the bases the following expressions for $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{R}$ can be given [185,190]

$$\mathbf{U} = \mathbf{s}_\alpha \otimes \mathbf{a}^\alpha + \mathbf{n} \otimes \mathbf{n} \ , \qquad \mathbf{U}^{-1} = \mathbf{a}_\alpha \otimes \mathbf{s}^\alpha + \mathbf{n} \otimes \mathbf{n} \ ,$$

$$\mathbf{V} = \bar{\mathbf{a}}_\alpha \otimes \mathbf{r}^\alpha + \bar{\mathbf{n}} \otimes \bar{\mathbf{n}} \ , \qquad \mathbf{V}^{-1} = \mathbf{r}_\alpha \otimes \bar{\mathbf{a}}^\alpha + \bar{\mathbf{n}} \otimes \bar{\mathbf{n}} \ , \tag{5.4}$$

$$\mathbf{R} = \bar{\mathbf{a}}_\alpha \otimes \mathbf{s}^\alpha + \bar{\mathbf{n}} \otimes \mathbf{n} = \mathbf{r}_\alpha \otimes \mathbf{a}^\alpha + \bar{\mathbf{n}} \otimes \mathbf{n} \ .$$

Any rotation tensor $\mathbf{R}$ may be represented by

$$R = \cos\omega 1 + \sin\omega \ e \times 1 + (1 - \cos\omega)e \times e . \tag{5.5}$$

where the unit vector e describes the axis of rotation of R and ω is the angle of rotation of R about the axis of rotation.

Sometime it is more convenient to describe rotations by means of an equivalent finite rotation vector, direction of which is e and the length is a function of ω . For example, the finite rotation vector $\Omega \equiv \sin\omega e$ was used in [247,248,178,73,184-186,190-194], the vector $\theta \equiv 2tg\frac{\omega}{2}e$ was used in [241,133,19] while $\omega \equiv \omega e$ was applied in [240]. As it was pointed out in [199], each of the definitions has some advantages: Ω is particularly convenient to be expressed in terms of displacements (cf. [185,192]), θ leads to geometric relations which do not contain trigonometric expressions while ω is the single-valued function of ω and can be defined in terms of the natural logarithm of R , cf. [199]. In [108,109] the rotations were described in terms of four Rodrigues parameters. In the following part of this report we shall use primarily the finite rotation vector θ , in terms of which transformation rules for the basic vectors are

$$\bar{a}_\beta = s_\beta + \frac{1}{t} \theta \times (s_\beta + \frac{1}{2} \theta \times s_\beta) , \quad t = 1 + \frac{1}{4} \theta \cdot \theta ,$$

$$r_\beta = a_\beta + \frac{1}{t} \theta \times (a_\beta + \frac{1}{2} \theta \times a_\beta) , \tag{5.6}$$

$$\bar{n} = Rn = n + \frac{1}{t} \theta \times (n + \frac{1}{2} \theta \times n) .$$

Let us introduce the relative (symmetric) surface stretch tensors

$$\eta = U - 1 = \eta_\beta \otimes a^\beta , \quad \eta_\beta = s_\beta - a_\beta = \eta_{\alpha\beta}a^\alpha ,$$

$$\varepsilon = V - 1 = \varepsilon_\beta \otimes r^\beta , \quad \varepsilon_\beta = \bar{a}_\beta - r_\beta = \eta_{\alpha\beta}r^\alpha , \tag{5.7}$$

$$\varepsilon = R\eta R^T , \quad \varepsilon_\beta = R\eta_\beta . \tag{5.8}$$

In terms of so defined $\eta_{\alpha\beta}$ many useful geometric relations may be derived, [185,190,193,199,19].

The corresponding relative (unsymmetric) surface bending tensors are defined by

$$\mu = (R^T \bar{n}_{,\beta} - n_{,\beta}) \otimes a^\beta = \mu_\beta \otimes a^\beta \ , \quad \mu_\beta = \mu_{\alpha\beta} a^\alpha \ ,$$

(5.9)

$$\lambda = (\bar{n}_{,\beta} - Rn_{,\beta}) \otimes r^\beta = \lambda_\beta \otimes r^\beta \ , \quad \lambda_\beta = \mu_{\alpha\beta} r^\alpha \ ,$$

$$\lambda = R \mu R^T \ , \quad \lambda_\beta = R \mu_\beta \ .$$

(5.10)

The relative surface strain measures $\eta_{\alpha\beta}$, $\mu_{\alpha\beta}$ were introduced first by ALUMÄE [5]. They are related to the Lagrangian surface strain measures (2.7) and (2.8) by

$$\gamma_{\alpha\beta} = \eta_{\alpha\beta} + \frac{1}{2}\eta_\alpha^\lambda \eta_{\lambda\beta} \ ,$$

(5.11)

$$\varkappa_{\alpha\beta} = \frac{1}{2} [(\delta_\alpha^\lambda + \eta_\alpha^\lambda)\mu_{\lambda\beta} + (\delta_\beta^\lambda + \eta_\beta^\lambda)\mu_{\lambda\alpha}] - \frac{1}{2}(b_\alpha^\lambda \eta_{\lambda\beta} + b_\beta^\lambda \eta_{\lambda\alpha}) \ .$$

Since $R^T R_{,\beta}$ and $R_{,\beta} R^T$ are scew-symmetric they are expressible, according to [199], by their respective axial vectors k_β and l_β , called also the vectors of change of curvature of the coordinate lines [232,190], by the relations

$$R^T R_{,\beta} = k_\beta \times 1 \ , \quad R_{,\beta} R^T = l_\beta \times 1 \ , \quad l_\beta = Rk_\beta \ .$$

(5.12)

Then from (5.6), (5.11) and (5.12) we obtain

$$\mu_\beta = k_\beta \times n \ , \quad \lambda_\beta = l_\beta \times \bar{n} \ ,$$

(5.13)

$$k_\beta = \varepsilon^{\alpha\lambda} \mu_{\alpha\beta} a_\lambda + k_\beta n = \frac{1}{t} (\theta_{,\beta} + \frac{1}{2}\theta_{,\beta} \times \theta) \ ,$$

(5.14)

$$l_\beta = \varepsilon^{\alpha\lambda} \mu_{\alpha\beta} r_\lambda + k_\beta \bar{n} = \frac{1}{t} (\theta_{,\beta} - \frac{1}{2}\theta_{,\beta} \times \theta) \ .$$

In components in the reference basis we have

$$\theta = \varepsilon^{\alpha\beta}\theta_\beta a_\alpha + \theta_3 n = \varepsilon^{\alpha\beta}\theta_\beta r_\alpha + \theta_3 \bar{n} \ , \tag{5.15}$$

$$r_\beta = r_{\lambda\beta} a^\lambda + r_\beta n \ , \qquad \theta_{,\beta} = \phi^\lambda_{\cdot\beta} a_\lambda + \phi_\beta n \ ,$$

$$r_{\lambda\beta} = a_{\lambda\beta} - \frac{1}{t}\left(\varepsilon_{\lambda\beta}\theta_3 + \frac{1}{2}\theta_\lambda\theta_\beta + \frac{1}{2}a_{\lambda\beta}\theta_3^2\right) \ ,$$

$$r_\beta = \frac{1}{t}\left(\theta_\beta + \frac{1}{2}\varepsilon_{\beta\alpha}\theta^\alpha\theta_3\right) \ . \tag{5.16}$$

$$\phi^\lambda_{\cdot\beta} = \varepsilon^{\lambda\varkappa}\theta_{\varkappa|\beta} - b^\lambda_\beta\theta_3 \ , \qquad \phi_\beta = \theta_{3,\beta} + b^\lambda_\beta\varepsilon_{\lambda\varkappa}\theta^\varkappa \ ,$$

$$n^\lambda = -\frac{1}{t}\left(\theta^\lambda + \frac{1}{2}\varepsilon^{\alpha\lambda}\theta_\alpha\theta_3\right) \ , \qquad n = 1 - \frac{1}{2t}\theta^\lambda\theta_\lambda \ . \tag{5.18}$$

Using (5.15) – (5.17) together with (2.2), (5.7)$_1$, (5.9)$_1$ and (5.14)$_1$ the relative symmetric surface strain measures may be expressed explicitly in terms of components of **u** and **θ** according to

$$\eta_{\alpha\beta} = r_\alpha \cdot \bar{a}_\beta - a_{\alpha\beta} = r_{\lambda\alpha} l^\lambda_{\cdot\beta} + r_\alpha \varphi_\beta - a_{\alpha\beta} =$$

$$= u_{\alpha|\beta} - b_{\alpha\beta}w - \frac{1}{1+\frac{1}{4}(\theta^\lambda\theta_\lambda + \theta_3^2)}\left[\varepsilon_{\alpha\beta}\theta_3 + \frac{1}{2}\theta_\alpha\theta_\beta + \frac{1}{2}a_{\alpha\beta}\theta_3^2 + \right.$$

$$\left. + (\varepsilon_{\alpha\lambda}\theta_3 + \frac{1}{2}\theta_\lambda\theta_\alpha + \frac{1}{2}a_{\lambda\alpha}\theta_3^2)(u^\lambda_{|\beta} - b^\lambda_\beta w) - (\theta_\alpha + \frac{1}{2}\varepsilon_{\alpha\varkappa}\theta^\varkappa\theta_3)(w_{,\beta} + b^\lambda_\beta u_\lambda)\right] \ , \tag{5.19}$$

$$\rho_{\alpha\beta} = \frac{1}{2}(\varepsilon_{\alpha\lambda}k_\beta + \varepsilon_{\beta\lambda}k_\alpha)\cdot a^\lambda = \frac{1}{2}(\mu_{\alpha\beta} + \mu_{\beta\alpha}) =$$

$$= \frac{1}{2t}\left[\varepsilon_{\alpha\lambda}(\phi^\lambda_{\cdot\beta} + \frac{1}{2}\theta^\lambda\phi_\beta) + \varepsilon_{\beta\lambda}(\phi^\lambda_{\cdot\alpha} + \frac{1}{2}\theta^\lambda\phi_\alpha) - \frac{1}{2}(\phi_{\alpha\beta} + \phi_{\beta\alpha})\theta_3\right] = \tag{5.20}$$

$$= -\frac{1}{2+\frac{1}{2}(\theta^\lambda\theta_\lambda + \theta_3^2)}\left[\theta_{\alpha|\beta} + \theta_{\beta|\alpha} + \varepsilon_{\alpha\lambda}(b^\lambda_\beta\theta_3 - \frac{1}{2}\theta^\lambda\theta_{3,\beta} + \frac{1}{2}\theta^\lambda_{|\beta}\theta_3)\right] \ .$$

Corresponding expressions for $\mu_{\alpha\beta}$ and k_β in terms of components of **θ** follow directly from (5.14)$_1$ and (5.15)$_1$. In components of **Ω** the relations (5.20) were given first by SIMMONDS and DANIELSON [247], while (5.19) by the author [184,185]. Equivalent relations for $\eta_{\alpha\beta}$, $\mu_{\alpha\beta}$ and k_β in terms of components of

ω were given by SHKUTIN [240]. Linearization of $\rho_{\alpha\beta}$ given by (5.20) with respect to displacements leads to the tensor of change of curvature which, according to BUDIANSKY and SANDERS [37], is the best choice for the linear theory of shells.

Rules of differentiation of the intermediate bases may be given with the help of (5.12) in the form [190,199]

$$s_{\alpha}||_{\beta} = - k_{\beta} \times s_{\alpha} + \bar{b}_{\alpha\beta}n \ , \quad n_{,\beta} = - k_{\beta} \times n - \bar{b}_{\beta}^{\alpha}s_{\alpha} \ ,$$

(5.21)

$$r_{\alpha}|_{\beta} = l_{\beta} \times r_{\alpha} + b_{\alpha\beta}\bar{n} \ , \quad \bar{n}_{,\beta} = l_{\beta} \times \bar{n} - b_{\beta}^{\alpha}r_{\alpha} \ .$$

Since $\bar{a}_{\alpha} = a_{\alpha} + u_{,\alpha}$ we can solve (5.6) and (5.14) for $u_{,\alpha}$ and $\theta_{,\alpha}$, what leads to

$$u_{,\alpha} = \eta_{\alpha} + \frac{1}{t} \theta \times (s_{\alpha} + \frac{1}{2} \theta \times s_{\alpha}) = \epsilon_{\alpha} - \frac{1}{t} \theta \times (r_{\alpha} - \frac{1}{2} \theta \times r_{\alpha}) \ ,$$

(5.22)

$$\theta_{,\alpha} = k_{\alpha} + \frac{1}{t} \theta \times k_{\alpha} + \frac{1}{4}(\theta \cdot k_{\alpha})\theta = l_{\alpha} - \frac{1}{t} \theta \times l_{\alpha} + \frac{1}{4}(\theta \cdot l_{\alpha})\theta \ .$$

Integrability conditions $\epsilon^{\alpha\beta}u_{,\alpha\beta} = 0$ and $\epsilon^{\alpha\beta}\theta_{,\alpha\beta} = 0$ of (5.22) give us the following two sets of vector equations

$$\epsilon^{\alpha\beta}(\eta_{\alpha}|_{\beta} + k_{\beta} \times s_{\alpha}) = 0 \ , \quad \epsilon^{\alpha\beta}(k_{\alpha}|_{\beta} + \frac{1}{2} k_{\beta} \times k_{\alpha}) = 0 \ ,$$

(5.23)

$$\epsilon^{\alpha\beta}(\epsilon_{\alpha}|_{\beta} + l_{\beta} \times r_{\alpha}) = 0 \ , \quad \epsilon^{\alpha\beta}(l_{\alpha}|_{\beta} + \frac{1}{2} l_{\alpha} \times l_{\beta}) = 0 \ .$$

(5.24)

These two sets of equations constitute two alternative vector forms of compatibility conditions in the non-linear theory of thin shells. The second equation of (5.23) was derived independently by CHERNYKH and SHAMINA [43] and PIETRASZKIEWICZ [184]. The vector equations (5.24) were derived first by SHKUTIN [239,240] and independently by AXELRAD [16] and LIBAI and SIMMONDS [133]. In component form the relations (5.24) were given already by ALUMÄE [5,7] and in orthogonal coordinates by REISSNER [208]. Since $\epsilon_{\beta} = Rn_{\beta}$ and $l_{\beta} = Rk_{\beta}$, both sets of compatibility conditions are transformable to each other. Several other

equivalent vector or tensor forms of compatibility conditions may also be constructed from the ones given by PIETRASZKIEWICZ and BADUR [199,200] for the three-dimensional deformation of a continuum. The three-dimensional compatibility conditions of [199,200] should be written on the reference surfaces $\mathcal{M}$ or $\mathcal{R}$ and Kirchhoff-Love constraints should be taken into account.

Within the K-L type shell theory finite rotations are expressible in terms of displacements by non-rational relations [185,190,192], expressed in the stretched basis

$$R = \bar{a}_\alpha \otimes s^\alpha + \bar{n} \otimes n = \bar{a}^{\alpha\beta}(\delta_\beta^\lambda + \eta_\beta^\lambda)(a_\alpha + u_{,\alpha}) \otimes a_\lambda + (n^\alpha a_\alpha + nn) \otimes n ,$$

$$\Omega = \frac{1}{2}(s_\alpha \times \bar{a}^\alpha + n \times \bar{n}) = \frac{1}{2}\bar{\varepsilon}^{\alpha\beta}[(\bar{n} \cdot s_\alpha - \varphi_\alpha)s_\beta + (u_{,\alpha} \cdot s_\beta)n] = \qquad (5.25)$$

$$= \frac{1}{2}\varepsilon_{\alpha\beta}\{[n^\alpha - \bar{a}^{\lambda\mu}(\delta_\lambda^\alpha + \eta_\lambda^\alpha)\varphi_\mu]a^\beta + \bar{a}^{\lambda\mu}(\delta_\lambda^\alpha + \eta_\lambda^\alpha)1_{,\mu}^\beta n\} ,$$

or in the rotated basis

$$R = r_\alpha \otimes a^\alpha + n \otimes \bar{n} = r^\alpha \otimes [(\delta_\alpha^\lambda + \eta_\alpha^\lambda)r_\lambda - u_{,\alpha}] + \bar{n} \otimes [\varphi_\alpha \bar{a}^{\alpha\beta}(\delta_\beta^\lambda + \eta_\beta^\lambda)r_\lambda + n\bar{n}] ,$$

$$\Omega = \frac{1}{2}(a_\alpha \times r^\alpha + n \times \bar{n}) = \frac{1}{2}\bar{\varepsilon}^{\alpha\beta}[(u_{,\alpha} \cdot r_\beta)\bar{n} + (n_\alpha - n \cdot r_\alpha)r_\beta] = \qquad (5.26)$$

$$= \frac{1}{2}\varepsilon_{\alpha\beta}\{[n^\alpha - \bar{a}^{\lambda\mu}(\delta_\lambda^\alpha + \eta_\lambda^\alpha)\varphi_\mu]r^\beta + \bar{a}^{\lambda\mu}(\delta_\lambda^\alpha + \eta_\lambda^\alpha)1_{,\mu}^\beta \bar{n}\} .$$

The dependency of rotations upon displacements can be also expressed implicitly, in the form of three constraint conditions [19]

$$n \cdot n_\beta = \bar{n} \cdot \varepsilon_\beta = (Rn) \cdot (a_\beta + u_{,\beta}) =$$

$$= n_\lambda 1_{,\beta}^\lambda + n\varphi_\beta = 0 ,$$

$$\qquad (5.27)$$

$$\varepsilon^{\alpha\beta}n_{\alpha\beta} = \varepsilon^{\alpha\beta}a_\alpha \cdot n_\beta = \varepsilon^{\alpha\beta}r_\alpha \cdot \varepsilon_\beta = \varepsilon^{\alpha\beta}(Ra_\alpha) \cdot (a_\beta + u_{,\beta} - Ra_\beta) =$$

$$= \varepsilon^{\alpha\beta}(r_{\lambda\alpha}1_{,\beta}^\lambda + r_\alpha\varphi_\beta) = 0 ,$$

where n_λ, n, $r_{\lambda\alpha}$, r_α are given in terms of rotation components by (5.16), (5.17) while $l^\lambda_{\cdot\alpha}$, φ_β are expressed in terms of displacement components by $(2.3)_1$.

5.2. Decomposition of deformation at the boundary

During the shell deformation the orthogonal triad $\mathbf{v}$, $\mathbf{t}$, $\mathbf{n}$ of $\mathcal{C}$ transforms into the orthonormal triad $\bar{\mathbf{a}}_v$, $\bar{\mathbf{a}}_t$, $\bar{\mathbf{n}}$ of $\bar{\mathcal{C}}$, where $\bar{\mathbf{a}}_v = \bar{\mathbf{a}}_t \times \bar{\mathbf{n}}$. According to the polar decomposition (5.1) we obtain

$$\bar{\mathbf{a}}_t = \bar{\mathbf{a}}_\alpha t^\alpha = \mathbf{R}\mathbf{s}_t = \mathbf{V}\mathbf{r}_t , \qquad \mathbf{s}_t = \mathbf{U}\mathbf{t}, \qquad \mathbf{r}_t = \mathbf{r}_\alpha t^\alpha = \mathbf{R}\mathbf{t} ,$$

$$(5.28)$$

$$\bar{\mathbf{a}}_v = \mathbf{R}\mathbf{s}_v = \mathbf{V}\mathbf{r}_v , \qquad\qquad \mathbf{s}_v = \mathbf{U}\mathbf{v}, \qquad \mathbf{r}_v = \mathbf{r}_\alpha t^\alpha = \mathbf{R}\mathbf{v} .$$

Since $\mathbf{v}$ and $\mathbf{t}$ do not coincide, in general, with the principal directions of $\mathbf{U}$, the action of $\mathbf{U}$ on $\mathbf{v}$ and $\mathbf{t}$ consists of an extension by a factor $\bar{\mathbf{a}}_t$ and a finite rotation about $\mathbf{n}$. This rotation may be described by the proper orthogonal tensor $\mathbf{Q}_U$. Similarly, the action of $\mathbf{V}$ on $\mathbf{r}_v$ and $\mathbf{r}_t$ consists of an extension by a factor $\bar{\mathbf{a}}_t$ and the finite rotation performed with the help of the proper orthogonal tensor $\mathbf{Q}_V$. Both rotations are defined by

$$\mathbf{Q}_U = \frac{1}{\bar{\mathbf{a}}_t} (\mathbf{s}_v \otimes \mathbf{v} + \mathbf{s}_t \otimes \mathbf{t}) + \mathbf{n} \otimes \mathbf{n} ,$$

$$(5.29)$$

$$\mathbf{Q}_V = \frac{1}{\bar{\mathbf{a}}_t} (\bar{\mathbf{a}}_v \otimes \mathbf{r}_v + \bar{\mathbf{a}}_t \otimes \mathbf{r}_t) + \bar{\mathbf{n}} \otimes \bar{\mathbf{n}} .$$

It is convenient to replace two subsequent rotations performed by $\mathbf{Q}_U$ and $\mathbf{R}$, or $\mathbf{R}$ and $\mathbf{Q}_V$, by one total rotation performed by the proper orthogonal tensor

$$\mathbf{R}_t = \mathbf{R}\mathbf{Q}_U = \mathbf{Q}_V\mathbf{R} = \frac{1}{\bar{\mathbf{a}}_t} (\bar{\mathbf{a}}_v \otimes \mathbf{v} + \bar{\mathbf{a}}_t \otimes \mathbf{t}) + \bar{\mathbf{n}} \otimes \mathbf{n} ,$$

$$(5.30)$$

$$\bar{\mathbf{a}}_v = \bar{\mathbf{a}}_t\mathbf{R}_t\mathbf{v} , \qquad \bar{\mathbf{a}}_t = \bar{\mathbf{a}}_t\mathbf{R}_t\mathbf{t} , \qquad \bar{\mathbf{n}} = \mathbf{R}_t\mathbf{n} .$$

Since $\mathbf{R}_t^T\mathbf{R}_t'$ and $\mathbf{R}_t'\mathbf{R}_t^T$ are scew-symmetric along $\mathcal{C}$, they are expressible in

terms of their respective axial vectors k_t and l_t, called the vectors of change of curvature of the boundary contour [178,185], by the relations (cf. [199])

$$R_t^T R_t' = k_t \times 1 , \quad R_t' R_t^T = l_t \times 1 , \quad l_t = R_t k_t . \tag{5.31}$$

Now derivatives of $\bar{a}_t$ and $\bar{n}$ along $\mathcal{C}$ can be given by

$$\bar{a}_t' = \bar{a}_t R_t [\frac{1}{\bar{a}_t^2} \gamma_{tt}' t + (\rho_t + k_t) \times t] , \tag{5.32}$$

$$\bar{n}' = R_t [(\rho_t + k_t) \times n] ,$$

where in components in the reference basis

$$\rho_t = \sigma_t \nu + \tau_t t + \varkappa_t n , \tag{5.33}$$

$$k_t = - k_{tt} \nu + k_{\nu t} t - k_{nt} n .$$

Exact expressions for components of k_t in $(5.33)_2$ were given by NOVOZHILOV and SHAMINA [178] and the author [184,185]. In terms of physical components of the Lagrangian strain measures on $\mathcal{C}$ these expressions are [193]

$$k_{tt} = \frac{1}{\bar{a}_t} [\sigma_t (\bar{a}_t - 1) + \varkappa_{tt}] ,$$

$$k_{\nu t} = \sqrt{\frac{a}{\bar{a}}} [\bar{a}_t (\tau_t + \varkappa_{\nu t}) + \frac{1}{\bar{a}_t} 2\gamma_{\nu t} (\sigma_t - \varkappa_{tt})] - \tau_t , \tag{5.34}$$

$$k_{nt} = \varkappa_t (1 - \frac{1}{\bar{a}_t^2} \sqrt{\frac{\bar{a}}{a}}) - \frac{1}{\bar{a}_t^2} \sqrt{\frac{a}{\bar{a}}} 2\gamma_{\nu t} (\gamma_{tt}' + 2\varkappa_t \gamma_{\nu t}) +$$

$$+ \sqrt{\frac{a}{\bar{a}}} [2\gamma_{\nu t}' - (\gamma_{tt})_{,\nu} + 2\varkappa_\nu \gamma_{\nu t} + 2\varkappa_t (\gamma_{\nu \nu} - \gamma_{tt})] .$$

Using (5.11) it is easy to express the components of k_t also in terms of physical components of the relative surface strain measures $\eta_{\alpha\beta}$ and $\mu_{\alpha\beta}$ on $\mathcal{C}$

During the shell deformation compatible with the K-L constraints the shell boundary surface $p(s,\zeta) = r(s) + \zeta n(s)$ deforms itself into the surface $\bar{p}(s,\zeta) = \bar{r}(s) + \bar{n}(s)$. According to discussion presented in the section 4.5 the boundary surface $\bar{p}(s,\zeta)$ may be entirely described by assuming $\bar{r} = \bar{r}^*$ and $\phi = \phi^*$ along $\mathcal{C}_u$. These conditions constitute the basic (displacement) version of geometric boundary conditions for the non-linear theory of shells.

The deformed boundary surface may also be described by the following differential equations [178]

$$\bar{p}_{,s} = \bar{r}' + \zeta\bar{n}' , \quad \bar{p}_{,\zeta} = \bar{n} , \quad \bar{r}' = \bar{a}_t , \tag{5.35}$$

$$\bar{p}_{,ss} = \bar{r}'' + \zeta\bar{n}'' , \quad \bar{p}_{,\zeta s} = \bar{n}' , \quad \bar{r}'' = \bar{a}'_t . \tag{5.36}$$

The set of equations (5.35) describes the surface $\bar{p}(s,\zeta)$ implicitly, with the accuracy up to a translation in the space. According to $(5.30)_2$ the right-hand sides of (5.35) are established if γ_{tt} and R_t are given along $\mathcal{C}_u$. The geometric conditions $\gamma_{tt} = \gamma_{tt}^*$, $R_t = R_t^*$ on $\mathcal{C}_u$ are called the kinematic boundary conditions of the non-linear theory of shells.

Also the set of equations (5.36) describes the surface $\bar{p}(s,\zeta)$ implicitly, with the accuracy up to a translation and rotation in the space. According to (5.32) the right-hand sides of (5.36) are established if γ_{tt}, k_t and R_t are given along $\mathcal{C}_u$. However, since R_t can always be included into the description of the arbitrary rotation in the space, it is enough to assume only γ_{tt} and k_t to be given on $\mathcal{C}_u$. The geometric conditions $\gamma_{tt} = \gamma_{tt}^*$, $k_t = k_t^*$ on $\mathcal{C}_u$ are called the deformational boundary conditions of the non-linear theory of shells.

In the case of the geometrically non-linear theory of shells we can simplify the components of k_t given in (5.34) by omitting small strains with respect to the unity, what leads to [190,192]

$$k_{tt} \approx \varkappa_{tt} + \sigma_t \gamma_{tt} ,$$

$$k_{vt} \approx \varkappa_{vt} + 2(\sigma_t - \varkappa_{tt})\gamma_{vt} - \tau_t \gamma_{vv} , \tag{5.37}$$

$$k_{nt} \approx 2\gamma'_{\nu t} - \gamma_{tt,\nu} + 2\varkappa_\nu \gamma_{\nu t} - \varkappa_t (\gamma_{tt} - \gamma_{\nu\nu}) \ .$$

In terms of the relative strain measures these approximate relations are

$$k_{tt} \approx \gamma_{tt} + (\tau_t + \rho_{\nu t})\eta_{\nu t} \ ,$$

$$k_{\nu t} \approx \rho_{\nu t} + \frac{1}{2}\tau_t (\eta_{tt} - \eta_{\nu\nu}) + \frac{1}{2}(3\sigma_t - 5\rho_{tt})\eta_{\nu t} - \frac{1}{2}(\sigma_\nu - \rho_{\nu\nu})\eta_{\nu t} \ , \tag{5.38}$$

$$k_{nt} \approx 2\eta'_{\nu t} - \eta_{tt,\nu} + 2\varkappa_\nu \eta_{\nu t} - \varkappa_t (\eta_{tt} - \eta_{\nu\nu}) \ .$$

These results were extended recently [41] to the large-strain theory of shells.

5.3. Shell equations in the rotated basis

Let us introduce the expressions (5.11) into the principle of virtual dis-placements (3.12), what gives

$$\iint\limits_{\mathcal{M}} (S^{\alpha\beta}\delta\eta_{\alpha\beta} + G^{\alpha\beta}\delta\rho_{\alpha\beta})dA =$$

$$\tag{5.39}$$

$$= \iint\limits_{\mathcal{M}} (\mathbf{p}\cdot\delta\mathbf{u} + \mathbf{h}\cdot\delta\boldsymbol{\beta})dA + \int\limits_{\mathcal{C}_f} (\mathbf{T}\cdot\delta\mathbf{u} + \mathbf{H}\cdot\delta\boldsymbol{\beta})ds \ ,$$

where the following stress and strain measures have been used

$$S^{\alpha\beta} = N^{\alpha\beta} + \frac{1}{2} (\eta^\alpha_\lambda N^{\lambda\beta} + \eta^\beta_\lambda N^{\lambda\alpha}) + \frac{1}{2} [(\mu^{\alpha\lambda} - b^{\alpha\lambda})M^\beta_\lambda + (\mu^{\beta\lambda} - b^{\beta\lambda})M^\alpha_\lambda] \ ,$$

$$\tag{5.40}$$

$$G^{\alpha\beta} = M^{\alpha\beta} + \frac{1}{2} (\eta^\alpha_\lambda M^{\lambda\beta} + \eta^\beta_\lambda M^{\lambda\alpha}) \ ,$$

$$\mu_{\alpha\beta} = \rho_{\alpha\beta} + \frac{1}{2} \varepsilon_{\alpha\beta}\rho \ , \qquad \rho_{\alpha\beta} = \frac{1}{2} (\mu_{\alpha\beta} + \mu_{\beta\alpha}), \qquad \rho = \varepsilon^{\alpha\beta}\mu_{\alpha\beta} \ . \tag{5.41}$$

Note that both surface stress measures $S^{\alpha\beta}$, $G^{\alpha\beta}$ and both surface strain measures $\eta_{\alpha\beta}$, $\rho_{\alpha\beta}$ are symmetric here by definition. They have been introduced first by ALUMÀE [8] and independently by SIMMONDS and DANIELSON [248].

Since $\delta R R^T$ is scew-symmetric, we express it in terms of its axial vector $\delta\omega$ by [199,137]

$$\delta R R^T = - R \delta R^T = \delta\omega \times 1 \ , \qquad \delta\omega = \frac{1}{t} \ (\delta\theta - \frac{1}{2} \ \delta\theta \times \theta) \ , \qquad (5.42)$$

which together with $(5.3)_1$ and $(5.6)_3$ leads to

$$\delta r_\alpha = \delta\omega \times r_\alpha \ , \qquad \delta\beta = \delta\bar{n} = \delta\omega \times \bar{n} \ . \qquad (5.43)$$

Taking variations of ε_β and λ_β given in $(5.7)_2$ and $(5.9)_2$ and using (2.1), (5.42) and (5.43) after transformations we also obtain

$$\delta\eta_{\alpha\beta} r^\alpha = \delta u_{,\beta} + \bar{a}_\beta \times \delta\omega \ , \qquad \delta\mu_{\alpha\beta} r^\alpha = \delta\omega_{,\beta} \times \bar{n} \ . \qquad (5.44)$$

If we take variations of the constraint conditions (5.27) and use (5.44) the following relations for $\delta\omega$ in the rotated basis are established

$$r^\lambda \cdot \delta\omega = \frac{1}{d} \ \varepsilon^{\alpha\beta} (\delta_\alpha^\lambda + \eta_\alpha^\lambda) \bar{n} \cdot \delta u_{,\beta} \ ,$$

$$\qquad (5.45)$$

$$\bar{n} \cdot \delta\omega = \frac{1}{2 + \eta_\varkappa^\varkappa} \ \varepsilon^{\beta\alpha} r_\alpha \cdot \delta u_{,\beta} \ .$$

With the help of (5.43) - (5.45) the principle of virtual displacements can be transformed into

$$- \iint\limits_{\mathscr{M}} (\hat{N}^\beta|_\beta + p) \cdot \delta u dA + \sum_j (\hat{F}_j - \hat{F}_j^*) \cdot \delta u_j +$$

$$\qquad (5.46)$$

$$+ \int\limits_{\mathcal{C}_f} [(\hat{P} - \hat{P}^*) \cdot \delta u - (\hat{M} - \hat{M}^*) \bar{n} \cdot \delta u_{,\nu}] ds = 0 \ ,$$

where

$$\hat{N}^\beta = (S^{\alpha\beta} + \frac{1}{2}\epsilon^{\alpha\beta}S)r_\alpha + Q^\beta\bar{n} \; ,$$

$$S = \frac{2}{2+\eta^\alpha_\alpha}\epsilon_{\lambda\varkappa}[\eta^\lambda_\rho S^{\rho\varkappa} - (b^\lambda_\rho - \mu^\lambda_{\cdot\rho})G^{\rho\varkappa}] \; , \tag{5.47}$$

$$Q^\beta = \frac{1}{d}\epsilon^{\beta\rho}(\delta^\lambda_\rho + \eta^\lambda_\rho)[\epsilon_{\alpha\lambda}(G^{\alpha\varkappa}|_\varkappa + h\cdot r^\alpha) - G^\varkappa_\lambda k_\varkappa] \; ,$$

$$\hat{P} = \hat{N}^\beta\nu_\beta + \hat{F}' \; , \qquad \hat{P}^* = T + \hat{F}^{*'} \; ,$$

$$\hat{F} = \frac{1}{d}\nu^\alpha(\delta^\lambda_\alpha + \eta^\lambda_\alpha)\epsilon_{\lambda\varkappa}G^{\varkappa\beta}\nu_\beta\bar{n} \; , \qquad \hat{P}^* = \frac{1}{\bar{a}_t}(K_t - \frac{1}{d}2\gamma_{\nu t}K_\nu)\bar{n} \; , \tag{5.48}$$

$$\hat{M} = -\frac{1}{d}t^\alpha(\delta^\lambda_\alpha + \eta^\lambda_\alpha)\epsilon_{\lambda\varkappa}G^{\varkappa\beta}\nu_\beta \; , \qquad \hat{M}^* = \bar{a}_t\frac{1}{d}K_\nu \; .$$

For arbitrary δu, δu_j and $\bar{n}\cdot\delta u_{,\nu}$ from (5.46) follow the equilibrium equations and corresponding static boundary conditions

$$\hat{N}^\beta|_\beta + p = 0 \quad \text{in} \quad \mathcal{M} \; ,$$

$$\hat{P} = \hat{P}^* \; , \quad \hat{M} = \hat{M}^* \quad \text{on} \quad \mathcal{C}_f \; , \tag{5.49}$$

$$\hat{F}_j = \hat{F}^*_j \quad \text{at each corner} \quad M_j \in \mathcal{C}_f \; .$$

It was shown in [142] that $\bar{n}\cdot\delta u_{,\nu}$ is not integrable in terms of displacement derivatives on $\mathcal{C}$, i.e. there exists no displacement boundary conditions which would be work-conjugate to (5.49). In this chapter we shall use the relations (5.49) to derive the set of shell equations in terms of rotations and other field variables as independent variables. Therefore, there will be no need to use displacement boundary conditions. However, if one would like to discuss such work-conjugate static and geometric (displacement) boundary conditions, one should apply the general formula (4.30) to transform the corresponding boundary terms in (5.39). Then some modified static boundary parameters $\hat{P}$, $\hat{M}$, $\hat{F}$ could be

calculated to which there would correspond some work-conjugate displacement para-meters **u**, ϕ . In this way one could construct an alternative form of the Lagrangian shell equations written in terms of $S^{\alpha\beta}$, $G^{\alpha\beta}$, $\eta_{\alpha\beta}$, $\mu_{\alpha\beta}$, k_β as given functions of displacements and their derivatives. Here we are not interested in such alternative displacement shell equations.

The equilibrium equations $(5.49)_1$ can be presented in component form in the rotated basis $r_\alpha, \bar{n}$, what gives

$$S^{\alpha\beta}|_\beta - \varepsilon^{\alpha\lambda}S^\beta_\lambda k_\beta + \frac{1}{2}\varepsilon^{\alpha\beta}S_{,\beta} + \frac{1}{2}a^{\alpha\beta}Sk_\beta - Q^\beta(b^\alpha_\beta - \rho^\alpha_\beta - \frac{1}{2}a^{\alpha\lambda}\varepsilon_{\lambda\beta}\rho) + \hat{p}^\alpha = 0 ,$$

$$\tag{5.51}$$

$$S^{\alpha\beta}(b_{\alpha\beta} - \rho_{\alpha\beta}) - \frac{1}{2}S\rho + Q^\beta|_\beta + q = 0 ,$$

where $\hat{p}^\alpha = p \cdot r^\alpha$ and S, Q^β are functions of $S^{\alpha\beta}$, $G^{\alpha\beta}$ given in (5.47).

The dependence of rotations upon displacements has been explicitly taken here into account by applying (5.45) in the transformation of (5.39) into (5.46) and using in (5.46) the variational expression $\bar{n} \cdot \delta u_{,\nu}$. However, when we intend to use rotations as independent variables, the dependence of **R** upon **u** should be implicitly taken into account. According to [19] this implicit dependence can be given by three constraint conditions (5.27) for the relative stretch vector $\varepsilon_\beta = \eta_{\alpha\beta}r^\alpha$. In terms of variations these constraints in $\mathscr{M}$ are

$$\varepsilon^{\alpha\beta}r_\alpha \cdot \delta\eta_{\lambda\beta}r^\lambda = 0 , \qquad \bar{n} \cdot \delta\eta_{\alpha\beta}r^\alpha = 0 . \tag{5.52}$$

Let $\frac{1}{2}S$ and Q^β be Lagrange multipliers associated with the respective con-straints (5.27) and (5.52). Then the left-hand side of (5.39) can be presented in an alternative form

$$\iint_{\mathscr{M}} \{[S^{\alpha\beta} + \frac{1}{2}\varepsilon^{\alpha\beta}S)r_\alpha + Q^\beta\bar{n}] \cdot \delta\eta_{\lambda\beta}r^\lambda + G^{\alpha\beta}\delta\rho_{\alpha\beta}\}dA , \tag{5.53}$$

where $\delta\eta_{\alpha\beta}$ and $\delta\rho_{\alpha\beta}$ are given by (5.44) in terms of now independent δu and $\delta\omega$.

Similar constraint conditions (5.52) should also be applied at the shell

boundary, only then $(5.52)_2$ should be multiplied by t^β, what corresponds to the constraint $(2.15)_1$. If now A and B are Lagrange multipliers associated with the respective constraints (5.52) on $\mathcal{C}$ then we should add to the right-hand side of (5.39) the following line integral

$$\int_{\mathcal{C}_f} (A\varepsilon^{\alpha\beta} r_\alpha \cdot \delta\eta_{\lambda\beta} r^\lambda + B\bar{n} \cdot \delta\eta_{\alpha\beta} r^\alpha t^\beta)ds \ . \tag{5.54}$$

Now the principle of virtual displacements (5.39), with (5.53) as the left-hand side of (5.39) and (5.54) added to the right-hand side of (5.39), can be transformed with the help of (5.44) into

$$-\iint_{\mathcal{M}} [(\hat{N}^\beta|_\beta + p) \cdot \delta u + (\hat{M}^\beta|_\beta + \bar{a}_\beta \times \hat{N}^\beta + \bar{n} \times h) \cdot \delta\omega]dA +$$

$$+ \int_{\mathcal{C}_f} \{[\hat{N}^\beta \nu_\beta - T + (A r_\nu + B\bar{n})'] \cdot \delta u + [\hat{M}^\beta \nu_\beta - \bar{n} \times H + B\bar{a}_\nu - \tag{5.55}$$

$$- A(r_\nu \times \bar{a}_t - r_t \times \bar{a}_\beta \nu^\beta)] \cdot \delta\omega + A r_t \cdot \delta u_{,\nu}\}ds + \sum_j (A r_\nu + B\bar{n})_j \cdot \delta u_j = 0 \ ,$$

where $\hat{N}^\beta$ is given by $(5.47)_1$ in terms of $S^{\alpha\beta}$, S and Q^β as independent variables and $\hat{M}^\beta = \bar{n} \times G^{\alpha\beta} r_\alpha$.

It follows from (5.55) that for an arbitrary $r_t \cdot \delta u_{,\nu}$ on $\mathcal{C}_f$ we always have $A \equiv 0$, i.e. the constraint condition $(5.27)_2$ is always satisfied on $\mathcal{C}$. Taking this into account the line integral of (5.55) is reduced to

$$\int_{\mathcal{C}_f} \{[\hat{N}^\beta \nu_\beta - T + (B\bar{n})'] \cdot \delta u + [\hat{M}^\beta \nu_\beta - \bar{n} \times H + B\bar{a}_\nu] \cdot \delta\omega\}ds + \tag{5.56}$$

$$+ \sum_j (B\bar{n})_j \cdot \delta u_j \ .$$

Since δu and $\delta\omega \times \bar{n}$ are now independent, from (5.55) and (5.56) follow vector equilibrium equations and corresponding static boundary conditions

$$\tilde{N}^\beta|_\beta + p = 0 \ , \quad \tilde{M}^\beta|_\beta + \bar{a}_\beta \times \tilde{N}^\beta + \bar{n} \times h = 0 \quad \text{in} \ \mathcal{M} \ , \tag{5.57}$$

$$\tilde{N}^\beta \nu_\beta - T + (B\bar{n})' = 0 \ , \quad G^{\alpha\beta} r_\alpha \nu_\beta - H - B\bar{a}_t = 0 \quad \text{on} \ \mathcal{C}_f \ , \tag{5.58}$$

$$(B\bar{n})_j = 0 \quad \text{at each corner} \ M_j \ \epsilon \ \mathcal{C}_f \ . \tag{5.59}$$

Corresponding work-conjugate geometric boundary conditions follow from the kinematic constraints $\delta u = 0$, $\delta\omega \times \bar{n} = \delta\bar{n} = 0$ on $\mathcal{C}_u$ and $\delta u_i = 0$ on $M_i \ \epsilon \ \mathcal{C}_u$. For independent displacements and rotations these constraints have the solutions

$$u = u^* \ , \quad Rn = R^*n \quad \text{on} \ \mathcal{C}_u \ , \tag{5.60}$$

$$u_i = u_i^* \quad \text{at each corner} \ M_i \ \epsilon \ \mathcal{C}_u \ . \tag{5.61}$$

The second of (5.60) is still subjected to the two constraints: $(5.27)_1$ multiplied by t^β and $(5.27)_2$. Therefore, in fact $(5.60)_2$ describes implicitly only one scalar condition.

In components in the rotated basis $(5.57)_2$ takes the form

$$G^{\alpha\beta}|_\beta - \epsilon^{\alpha\lambda} G^\beta_\lambda k_\beta - (\delta^\alpha_\beta + \eta^\alpha_\beta) Q^\beta + \hbar^\alpha = 0 \ , \tag{5.62}$$

$$S(1 + \frac{1}{2}\eta^\alpha_\alpha) - \epsilon_{\alpha\lambda}\eta^\alpha_\beta S^{\lambda\beta} - \epsilon_{\alpha\lambda} G^{\alpha\beta}(b^\lambda_\beta - \rho^\lambda_\beta) - \frac{1}{2} G^\alpha_\alpha \rho = 0 \ ,$$

while the boundary conditions $(5.58)_2$ written relative to $\bar{\nu}, \bar{t}$ are

$$\frac{1}{\bar{a}_t} t_x(\delta^x_\alpha + \eta^x_\alpha) G^{\alpha\beta} \nu_\beta - K_t - B\bar{a}_t = 0 \ , \tag{5.63}$$

$$\frac{1}{\bar{a}_t} t^x(\delta^\lambda_x + \eta^\lambda_x) \epsilon_{\alpha\lambda} G^{\alpha\beta} \nu_\beta - K_\nu = 0 \ .$$

Note that only two components appear in (5.63), since A has been eliminated.

The equilibrium equations (5.51) and (5.62) were derived by ALUMÄE [8]. Equivalent forms of equilibrium equations are given in [5,248,240,19]. Boundary and corner conditions were not discussed in [5,8], while the four static boundary conditions derived in [248,19] would follow from our (5.58) after elimination of the Lagrange multiplier B with the help of $(5.63)_1$. But then it is not apparent how to construct the work-conjugate geometric boundary conditions corresponding to the constraint $\bar{\mathbf{v}} \cdot \delta \bar{\mathbf{n}} = 0$ on $\mathcal{C}_u$ used in [248] and to the equivalent constraint $\bar{\mathbf{t}} \cdot \delta \omega = 0$ used in [19]. Therefore, such work-conjugate geometric boundary conditions were not discussed in [248,19]. On the other hand, the kinematic parameter $\delta[\bar{\mathbf{t}} \cdot (\omega e)]$ used by SHKUTIN [240] cannot be regarded as to be equivalent to the one which would appear during the elimination of B from $(5.58)_2$. It seems that the choice of such a parameter in [240] resulted from an identification of the axial vector $\delta \omega$ defined by (5.42) with the variation of the finite rotation vector ωe , what is correct only for infinitesimal rotations.

5.4. Alternative shell equations in the undeformed basis

Sometime it may be more convenient to use an alternative form of non-linear shell equations discussed in section 5.3, which is referred entirely to the undeformed basis of $\mathcal{M}$. Having this in mind, let us introduce the axial vector $\delta \mathbf{w}$ of the scew-symmetric tensor $R^T \delta R$ in analogy to (5.42) by [199]

$$R^T \delta R = - \delta R^T R = \delta \mathbf{w} \times 1 \ , \quad \delta \mathbf{w} = \frac{1}{t} \left(\delta \theta + \frac{1}{2} \delta \theta \times \theta \right) \tag{5.64}$$

in terms of which variations of the relative strain measures $(5.7)_1$ and $(5.9)_1$ are given by

$$\delta \eta_\beta = \delta \eta_{\alpha\beta} a^\alpha = \delta \mathbf{v}_{,\beta} + k_\beta \times \delta \mathbf{v} + s_\beta \times \delta \mathbf{w} \ , \tag{5.56}$$

$$\delta \mu_\beta = \delta \mu_{\alpha\beta} a^\alpha = \delta \mathbf{w}_{,\beta} \times n + (k_\beta \times \delta \mathbf{w}) \times n \ , $$

$$\delta \mathbf{w} = R^T \delta \omega \ , \quad \delta \mathbf{v} = R^T \delta \mathbf{u} \ . \tag{5.66}$$

If the rotations are to be regarded as independent variables then the constraint conditions (5.52) are replaced by

$$\varepsilon^{\alpha\beta} a_\alpha \cdot \delta\eta_\beta = 0 \ , \quad n \cdot \delta\eta_\beta = 0 \quad \text{in} \quad \mathcal{M} \ ,$$

$$\varepsilon^{\alpha\beta} a_\alpha \cdot \delta\eta_\beta = 0 \ , \quad n \cdot \delta\eta_\beta t^\beta = 0 \quad \text{on} \quad \mathcal{C} \ . \tag{5.67}$$

Let again $\frac{1}{2} S$, Q^β are Lagrange multipliers associated with the respective constraints $(5.67)_1$ in $\mathcal{M}$ and A, B are Lagrangian multipliers associated with the respective constraints $(5.67)_2$ on $\mathcal{C}$. The constraint conditions $(5.67)_1$ multiplied by $\frac{1}{2} S$ and Q^β, respectively, may be introduced into the surface integral of the left-hand side of (5.39). Similarly, the constraint conditions $(5.67)_2$ multiplied by A and B, respectively, may be introduced into the line integral of the right-hand side of (5.39). Then so modified principle of virtual displacements can be transformed with the help of (5.65) and (5.66) into

$$-\iint_{\mathcal{M}} [(\tilde{N}^\beta|_\beta + k_\beta \times \tilde{N}^\beta + R^T p) \cdot \delta v + (\tilde{M}^\beta|_\beta + k_\beta \times \tilde{M}^\beta + s_\beta \times \tilde{N}^\beta + n \times R^T h) \cdot \delta w] dA +$$

$$+ \int_{\mathcal{C}_f} [(\tilde{N}^\beta v_\beta - R_t^T T + (Bn)' - Bn \times k_\beta t^\beta] \cdot \delta v \ ds + \tag{5.68}$$

$$+ \int_{\mathcal{C}_f} [\tilde{M}^\beta v_\beta - n \times R_t^T H - Bn \times s_\beta t^\beta] \cdot \delta w_t ds + \sum_j (Bn)_j \cdot \delta v_j = 0 \ ,$$

where now

$$\tilde{N}^\beta = (S^{\alpha\beta} + \frac{1}{2} \varepsilon^{\alpha\beta} S) a_\alpha + Q^\beta n = R^T \hat{N}^\beta \ ,$$

$$\tilde{M}^\beta = n \times G^{\alpha\beta} a_\alpha = R^T \hat{M}^\beta \ , \tag{5.69}$$

$$\delta w_t = R_t^T \delta \omega_t = \frac{1}{\bar{a}_t} t \times (\delta v' + k_\beta t^\beta \times \delta v) + (t \cdot \delta w_t) t \ .$$

In the transformations leading to (5.68) we have taken into account that $A \equiv 0$ on $\mathcal{C}_f$ for an arbitrary $t \cdot \delta v_{,\nu}$, in analogy to the reduction of (5.55) to (5.56).

Since δv and δw are independent, from (5.68) follow vector equilibrium equations and corresponding static boundary conditions

$$\tilde{N}^\beta\big|_\beta + k_\beta \times \tilde{N}^\beta + R^T p = 0 \ ,$$

$$\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \mathcal{M}, \qquad (5.70)$$

$$\tilde{M}^\beta\big|_\beta + k_\beta \times \tilde{M}^\beta + s_\beta \times \tilde{N}^\beta + n \times R^T h = 0 \ ,$$

$$\tilde{N}^\beta \nu_\beta - R_t^T T + (Bn)' - Bn \times k_\beta t^\beta = 0 \ ,$$

$$\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \mathcal{C}_f , \qquad (5.71)$$

$$G^{\alpha\beta} a_\alpha \nu_\beta - R_t^T H - B s_t = 0 \ ,$$

$$(Bn)_j = 0 \quad \text{at each corner} \quad M_j \ \epsilon \ \mathcal{C}_f \ . \qquad\qquad\qquad (5.72)$$

Component form of (5.70) in the undeformed basis a_α, n coincides with (5.51) and (5.62) while the components of (5.71) in the basis v, t are equivalent to those given in (5.63).

Alternative forms of equilibrium equations written in the stretched basis s_α, n are given by the author [185,190,193] and in the rotated basis by KAYUK and SAKHATZKII [109].

5.5. Static-geometric analogy

Between the equilibrium equations (5.57) and the compatibility conditions (5.24) (or, equivalently, between (5.70) and (5.24)) an interesting static-geometric analogy can be established. In order to show this analogy, let us express compatibility conditions (5.24) in component form in the rotated basis, what gives

$$\varepsilon^{\alpha\beta} n_{\lambda\alpha}|_{\beta} + \varepsilon^{\alpha\beta}(\delta^{\varkappa}_{\alpha} + n^{\varkappa}_{\alpha})\varepsilon_{\varkappa\lambda} k_{\beta} = 0 \ ,$$

$$\rho(1 + \tfrac{1}{2} n^{\varkappa}_{\varkappa}) - \varepsilon^{\alpha\beta} n^{\varkappa}_{\alpha}(b_{\varkappa\beta} - \rho_{\varkappa\beta}) = 0 \ ,$$

$$\varepsilon^{\alpha\beta}\varepsilon^{\lambda\varkappa} \rho_{\lambda\alpha}|_{\beta} + \tfrac{1}{2}\varepsilon^{\varkappa\beta}\rho_{,\beta} - \varepsilon^{\alpha\beta} k_{\alpha}(b^{\varkappa}_{\beta} - \rho^{\varkappa}_{\beta}) + \tfrac{1}{2} a^{\alpha\varkappa} k_{\alpha}\rho = 0 \ ,$$

$$\varepsilon^{\alpha\beta}\varepsilon^{\lambda\varkappa}(b_{\lambda\alpha} - \tfrac{1}{2}\rho_{\lambda\alpha})\rho_{\varkappa\beta} - \tfrac{1}{4}\rho^2 + \varepsilon^{\alpha\beta} k_{\alpha}|_{\beta} = 0 \ .$$

(5.73)

Let us introduce modified measures by the formal relations

$$\tilde{\rho}^{\alpha\beta} = - \varepsilon^{\alpha\sigma}\varepsilon^{\beta\tau}\rho_{\sigma\tau}, \quad \tilde{n}^{\alpha\beta} = + \varepsilon^{\alpha\sigma}\varepsilon^{\beta\tau} n_{\sigma\tau}, \quad \tilde{k}^{\alpha} = + \varepsilon^{\alpha\sigma} k_{\sigma}, \quad \tilde{\rho} = - \rho \ ,$$

(5.74)

$$\rho_{\alpha\beta} = - \varepsilon_{\alpha\sigma}\varepsilon_{\beta\tau}\tilde{\rho}^{\sigma\tau}, \quad n_{\alpha\beta} = + \varepsilon_{\alpha\sigma}\varepsilon_{\beta\tau}\tilde{n}^{\sigma\tau}, \quad k_{\alpha} = - \varepsilon_{\alpha\sigma}\tilde{k}^{\sigma}, \quad \rho = - \tilde{\rho} \ .$$

In terms of those modified measures the compatibility conditions (5.73) can be written in the form

$$\tilde{n}^{\alpha\beta}|_{\beta} - \tfrac{1}{2}\varepsilon^{\alpha\lambda}\tilde{n}^{\beta}_{\lambda} k_{\beta} - (\delta^{\alpha}_{\beta} + \tfrac{1}{2} n^{\alpha}_{\beta})\tilde{k}^{\beta} = 0 \ ,$$

$$\tilde{\rho}(1 + \tfrac{1}{4} n^{\alpha}_{\alpha}) - \tfrac{1}{2}\varepsilon_{\alpha\lambda} n^{\alpha}_{\beta}\tilde{\rho}^{\lambda\beta} - \varepsilon_{\alpha\lambda}\tilde{n}^{\alpha\beta}(b^{\lambda}_{\beta} - \tfrac{1}{2}\rho^{\lambda}_{\beta}) - \tfrac{1}{4}\tilde{n}^{\alpha}_{\alpha}\rho = 0 \ ,$$

(5.75)

$$\tilde{\rho}^{\alpha\beta}|_{\beta} - \tfrac{1}{2}\varepsilon^{\alpha\lambda}\tilde{\rho}^{\beta}_{\lambda} k_{\beta} + \tfrac{1}{2}\varepsilon^{\alpha\beta}\tilde{\rho}_{,\beta} + \tfrac{1}{4} a^{\alpha\beta}\tilde{\rho} k_{\beta} - \tilde{k}^{\beta}(b^{\alpha}_{\beta} - \tfrac{1}{2}\rho^{\alpha}_{\beta} - \tfrac{1}{4} a^{\alpha\lambda}\varepsilon_{\lambda\beta}\rho) = 0,$$

$$\tilde{\rho}^{\alpha\beta}(b_{\alpha\beta} - \tfrac{1}{2}\rho_{\alpha\beta}) - \tfrac{1}{4}\tilde{\rho}\rho + \tilde{k}^{\beta}|_{\beta} = 0 \ .$$

If we compare (5.75) with the equilibrium equations (5.62) and (5.51) we note that the homogeneous equilibrium equations can be transformed into the modified compatibility conditions (5.75) if $S^{\alpha\beta}$, $G^{\alpha\beta}$, Q^{β}, S are replaced by $\tilde{\rho}^{\alpha\beta}$, $\tilde{n}^{\alpha\beta}$, $\tilde{k}^{\beta}$, $\tilde{\rho}$, respectively, and all non-linear (quadratic) terms are multiplied by 1/2. This static-geometric analogy was noted by ALUMÄE [7,8]. It extends to the non-linear theory of shells the static-geometric analogy of the linear theory of shells which was formulated in tensor form by GOLDENVEIZER [76].

The compatibility conditions $(5.73)_{1,2}$ can always be solved for k_{α} and ρ ,

what gives

$$k_\alpha = -\sqrt{\frac{a}{\bar{a}}} \, \epsilon^{\rho\varkappa}(\delta_\alpha^\lambda + n_\alpha^\lambda)\eta_{\lambda\rho}|_\varkappa \, ,$$

$$\rho = \frac{1}{1 + \frac{1}{2}n_\varkappa^\varkappa} \, \epsilon^{\alpha\beta}n_\alpha^\lambda(b_{\lambda\beta} - \rho_{\lambda\beta}) \, . \tag{5.76}$$

Similarly, the equilibrium equations (5.63) can be solved for Q^β, S what gives the formulae (5.47)$_{2,3}$. Then k_α, ρ, Q^β and S may be eliminated from the remaining equilibrium equations (5.51) and the compatibility conditions (5.73)$_{3,4}$, which then are expressed entirely in terms of symmetric measures $S^{\alpha\beta}$, $G^{\alpha\beta}$ and $\eta_{\alpha\beta}$, $\rho_{\alpha\beta}$. Unfortunately, for such transformed set of 3+3 equations the static-geometric analogy formulated above does not hold.

5.6. Shell equations in terms of rotations, displacements and Lagrange multipliers

Various non-linear shell relations discussed in the preceding sections allow for some freedom in choosing independent field variables of an appropriate boundary value problem.

An interesting version of the non-linear theory of shells can be given in terms of finite rotations θ, displacements u and Lagrange multipliers S, Q^β as independent field variables [19].

In terms of corresponding stress and strain measures $S^{\alpha\beta}$, $G^{\alpha\beta}$ and $\eta_{\alpha\beta}$, $\rho_{\alpha\beta}$ the strain energy density (3.32) and the constitutive equations (3.34) are

$$\Sigma = \frac{h}{2} H^{\alpha\beta\lambda\mu}(\eta_{\alpha\beta}\eta_{\lambda\mu} + \frac{h^2}{12}\rho_{\alpha\beta}\rho_{\lambda\mu}) + O(Eh\eta^2\theta^2) \, , \tag{5.77}$$

$$S^{\alpha\beta} = \frac{\partial\Sigma}{\partial\rho_{\alpha\beta}} = C[(1-\nu)\eta^{\alpha\beta} + \nu a^{\alpha\beta}n_\varkappa^\varkappa] + O(Eh\eta\theta^2) \, , \quad C = \frac{Eh}{1-\nu^2} \, , \tag{5.78}$$

$$G^{\alpha\beta} = \frac{\partial\Sigma}{\partial\rho_{\alpha\beta}} = D[(1-\nu)\rho^{\alpha\beta} + \nu a^{\alpha\beta}\rho_\varkappa^\varkappa] + O(Eh^2\eta\theta^2), \quad D = \frac{Eh^3}{12(1-\nu^2)} \, .$$

Let the constitutive equations (5.78) together with (5.76) be introduced into

the equilibrium equations (5.51), (5.62) and then $\eta_{\alpha\beta}$, $\rho_{\alpha\beta}$ be expressed in terms of u, θ with the help of geometric relations (5.19) and (5.20). As a result, the problem is reduced to nine partial differential equations: six equilibrium equations (5.19), (5.20) expressed in terms of θ, u, S, Q^β and three constraint conditions (5.27) containing only θ, u . Corresponding work-conjugate static and geometric boundary and corner conditions are given by (5.58), (5.60) and (5.59), (5.61), respectively.

The structure of the final set of nine equations is relatively simple. The equilibrium equations (5.51), (5.62) are linear in S, Q^β and their first derivatives, are quadratic in $u_{,\beta}$ but linear in $u_{,\alpha\beta}$ while rotations appear in them as polynomials which are quadratic in $\theta_{,\beta}$ but again only linear in $\theta_{,\alpha\beta}$. The constraint conditions (5.27) are polynomials in rotations but linear in $u_{,\beta}$.

This system of nine non-linear equations may be considerably simplified in the case of small strains, when additionally we assume that the strains caused by stretching and bending of the reference surface are of comparable order, i.e. $\eta_{\alpha\beta} \sim h\rho_{\alpha\beta}$. Within the accuracy of the first-approximation theory from the compatibility conditions (5.73) we obtain the estimates

$$k_\beta = O(\eta/\lambda) , \quad \rho = O(\frac{\eta\theta^2}{h}), \quad \rho^\alpha_{\alpha|\beta} - \rho^\alpha_{\beta|\alpha} = O(\frac{\eta\theta^2}{h\lambda}) \tag{5.79}$$

which introduced together with (5.78) into (5.51), (5.62) lead to the following consistently approximated equilibrium equations

$$C[(1-\nu)\eta^\beta_\alpha|_\beta + \nu\eta^\beta_\beta|_\alpha] + \hat{p}^\alpha = O(Eh \frac{\eta\theta^2}{\lambda}) ,$$

$$C(b^\alpha_\beta - \rho^\alpha_\beta)[(1-\nu)\eta^\beta_\alpha + \nu\delta^\beta_\alpha\eta^\lambda_\lambda] + Q^\beta|_\beta + q = O(Eh^2 \frac{\eta\theta^2}{\lambda^2}) ,$$

$$D\rho^\beta_\beta|^\alpha - Q^\alpha + \hat{h}^\alpha = O(Eh^2 \frac{\eta\theta^2}{\lambda}) , \tag{5.80}$$

$$S - \varepsilon_{\lambda\beta}\{C\eta^\lambda_\alpha[(1-\nu)\eta^{\alpha\beta} + \nu a^{\alpha\beta}\eta^\varkappa_\varkappa] - D(b^\lambda_\alpha-\rho^\lambda_\alpha)[(1-\nu)\rho^{\alpha\beta} + \nu a^{\alpha\beta}\rho^\varkappa_\varkappa]\} = O(Eh^2\eta^2\theta^2) .$$

Within this approximation S apears only in the last algebraic equation $(5.80)_4$ and can be evaluated separately. The equation $(5.80)_3$ can also be solved

for Q^α and introduced into $(5.80)_2$, which then takes the form

$$D\rho_\alpha^{\ \alpha}\big|_\beta + C(b_\beta^\alpha - \rho_\beta^\alpha)[(1-\nu)\eta_\alpha^\beta + \nu\delta_\alpha^\beta n^\varkappa_\varkappa] + q + \hat{h}^\alpha\big|_\alpha = O(Eh^2 \frac{\eta\theta^2}{\lambda^2}) \ . \tag{5.81}$$

If now (5.19) and (5.20) are introduced into $(5.80)_1$ and (5.81) then we obtain

$$Ca^{\beta\rho}[(1-\nu)(r_{\lambda\alpha}1^\lambda_{\cdot\rho} + r_\alpha\varphi_\rho)\big|_\beta + \nu(r_{\lambda\beta}1^\lambda_{\cdot\rho} + r_\beta\varphi_\rho)\big|_\alpha] + \hat{p}_\alpha = O(Eh \frac{\eta\theta^2}{\lambda}) \ ,$$

$$D\{\frac{1}{t}[\varepsilon^{\alpha\lambda}(\psi_{\lambda\alpha} + \frac{1}{2}\theta_\lambda\psi_\alpha) - \frac{1}{2}\psi^\alpha_{\cdot\alpha}\theta_3]\}\big|_\beta^\beta +$$

$$+ C\{b_\beta^\alpha - \frac{1}{2t}a^{\alpha\varkappa}[\varepsilon_{\varkappa\gamma}(\psi^\gamma_{\cdot\beta} + \frac{1}{2}\theta^\gamma\psi_\beta) + \varepsilon_{\beta\gamma}(\psi^\gamma_{\cdot\varkappa} + \frac{1}{2}\theta^\gamma\psi_\varkappa) - \frac{1}{2}(\psi_{\varkappa\beta} + \psi_{\beta\varkappa})\theta_3]\} \times$$

$$\times [(1-\nu)a^{\beta\rho}(r_{\lambda\alpha}1^\lambda_{\cdot\rho} + r_\alpha\varphi_\rho) + \nu\delta_\alpha^\beta a^{\varkappa\rho}(r_{\lambda\varkappa}1^\lambda_{\cdot\rho} + r_\varkappa\varphi_\rho)] + q + \hat{h}^\alpha\big|_\alpha = O(Eh^2 \frac{\eta\theta^2}{\lambda^2}) \ . \tag{5.82}$$

The equations (5.82) together with (5.27) give us six partial differential equations for six components of θ, u to be solved.

Simplified static boundary and corner conditions on $\mathcal{C}_f$ follow from the corresponding reduction of (5.58) and (5.59), with the help of (5.78), (5.32) and (5.38). In the right-hand sides only the principal terms, which have the same structure as those in the approximate left-hand sides, are taken into account. As a result, we obtain on $\mathcal{C}_f$

$$C(\eta_{\nu\nu} + \nu n_{tt}) = Q_\nu + O(Eh\eta\theta^2) \ ,$$

$$C(1-\nu)n_{\nu t} = Q_t + O(Eh\eta\theta^2) \ ,$$

$$\tag{5.83}$$

$$D[(\rho_{\nu\nu} + \rho_{tt})_{,\nu} + (1-\nu)\rho_{\nu t}'] + \hat{h}_\nu = Q + K' + O(Eh^2 \frac{\eta\theta^2}{\lambda}) \ ,$$

$$D(\rho_{\nu\nu} + \nu\rho_{tt}) = K_\nu + O(Eh^2\eta\theta^2)$$

and $D(1-\nu)[\rho_{\nu t}\bar{n}]_j = [K_t\bar{n}]_j + O(Eh^2\eta\theta^2)$ on $M_j \in \mathcal{C}_f$, $\tag{5.84}$

where the relative strain measures still have to be expressed in terms of components of θ, u by (5.19) and (5.20). Corresponding work-conjugate geometric boundary and corner conditions are given in (5.60) and (5.61), with $(5.27)_1$ multiplied by t^β and $(5.27)_2$ as the constraints.

Let us assume that the external loads p, h, T and H are derivable from the potential functions $\Phi[u,\beta(R)]$ and $\psi[u,\beta(R)]$ by the relations (4.4). Note that now u and R may be treated as independent variables, what allows for some flexibility in the definition of the conservative loads. If the external loads are conservative the total potential energy of the shell is given by the functional

$$
I = \iint\limits_{\mathcal{M}} \{\Sigma[\eta_{\alpha\beta}(u,R),\rho_{\alpha\beta}(R)] + \frac{1}{2}\,\epsilon^{\alpha\beta}S\eta_{\alpha\beta}(u,R) +
$$

$$
+ Q^\beta nR^T\epsilon_\beta(u,R) + \Phi[u,\beta(R)]\}dA + \tag{5.85}
$$

$$
+ \int\limits_{\mathcal{C}_f} \{\psi[u,\beta(R)] - BnR^T\epsilon_\beta(u,R)t^\beta\}ds
$$

with the geometric boundary (5.60) and corner (5.61) conditions and the constraints (5.27) on $\mathcal{C}_u$ as subsidiary conditions. The variational principle $\delta I = 0$ states that among all possible values of independent fields u, R, S, Q^β and B, which are subjected to the conditions (5.60), (5.61) and (5.27) on $\mathcal{C}_u$, the actual solution renders the functional I stationary. The stationarity conditions of I are: the equilibrium equations (5.57) in $\mathcal{M}$, the constraint conditions (5.27) in $\mathcal{M}$, the static boundary and corner conditions (5.58), (5.59) on $\mathcal{C}_f$ and the constraint condition $\bar{n}\cdot\epsilon_\beta t^\beta = 0$ on $\mathcal{C}_f$.

Note that the functional I defined by (5.85) is linear in S, Q^β, B and is rational in u, R and their only **first** surface derivatives. The later property is important for the computerized numerical analysis of the flexible shells based on direct discretization of the functional (5.85). It allows to apply the simplest shape functions in the finite-element analysis or the simplest difference schemes in the finite-difference analysis, which assure high

efficiency of numerical algorithms and better convergence to the accurate final results.

In some applications it may be convenient to apply the more general free functional

$$
I_1 = \iint\limits_{\mathcal{M}} \{\Sigma(\eta_{\alpha\beta}, \rho_{\alpha\beta}) + \frac{1}{2} \epsilon^{\alpha\beta} S\eta_{\alpha\beta} + Q^\beta \bar{\mathbf{n}} \cdot \boldsymbol{\epsilon}_\beta -
$$

$$
- \frac{1}{2} S^{\alpha\beta} [\eta_{\alpha\beta} - \eta_{\alpha\beta}(\mathbf{u},\mathbf{R})] - G^{\alpha\beta} [\rho_{\alpha\beta} - \rho_{\alpha\beta}(\mathbf{R})] + \Phi[\mathbf{u}, \boldsymbol{\beta}(\mathbf{R})] \} dA +
$$

$$
(5.86)
$$

$$
+ \int\limits_{\mathcal{C}_f} \psi[\mathbf{u}, \boldsymbol{\beta}(\mathbf{R})] ds - \int\limits_{\mathcal{C}} B\mathbf{n}\mathbf{R}^T \cdot \boldsymbol{\epsilon}_\beta(\mathbf{u},\mathbf{R}) t^\beta ds - \sum_i (B\bar{\mathbf{n}})_i \cdot (\mathbf{u} - \mathbf{u}^*) -
$$

$$
- \int\limits_{\mathcal{C}_u} \{[\mathbf{N}^\beta \nu_\beta + (B\bar{\mathbf{n}})'] \cdot (\mathbf{u} - \mathbf{u}^*) + [G^{\alpha\beta} \mathbf{r}_\alpha \nu_\beta - B\bar{\mathbf{a}}_t] \cdot (\mathbf{R}\mathbf{n} - \mathbf{R}^* \bar{\mathbf{n}})\} ds .
$$

This free functional follows from (5.85) if we introduce into it the strain-displacement-rotation relations (5.27), the geometric boundary conditions (5.60) and the geometric corner conditions (5.61) multiplied by the respective Lagrange multipliers $P^{\alpha\beta}$, $K^{\alpha\beta}$, $\mathbf{P}$, $\mathbf{K}$, $\mathbf{S}_i$. Then some stationarity conditions of so defined I_1 allow to identify the Lagrange multipliers to be $S^{\alpha\beta}$, $G^{\alpha\beta}$, $\mathbf{N}^\beta \nu_\beta +$ $(B\bar{\mathbf{n}})'$, $G^{\alpha\beta} \mathbf{r}_\alpha \nu_\beta - B\bar{\mathbf{a}}_t$ and $(B\bar{\mathbf{n}})_i$, respectively, which have been already used in (5.86). The functional I_1 in (5.86) is defined on the following free fields subject to variation: $\mathbf{u}$, $\mathbf{R}$ in $\mathcal{M}$, $\mathbf{u}$, $\mathbf{R}$ on $\mathcal{C}$, $\mathbf{u}_i$ at each $M_i \in \mathcal{C}_u$; $\eta_{\alpha\beta}$, $\rho_{\alpha\beta}$, $S^{\alpha\beta}$, $G^{\alpha\beta}$, S, Q^β in $\mathcal{M}$, $S^{\alpha\beta}$, $G^{\alpha\beta}$, S, Q^β, B on $\mathcal{C}_u$, B on $\mathcal{C}_f$ and B_i on $M_i \in \mathcal{C}_u$. The variational principle $\delta I_1 = 0$ is equivalent to the complete set of non-linear shell equations: (5.57), (5.58), (5.59), (5.27), (5.19), (5.20), (5.60), (5.61) and (5.78).

5.7. Shell equations in terms of rotations and stress functions

If all the external forces are functions of the finite rotations alone the set of non-linear shell equations can be expressed in terms of the finite rotation vector θ and the stress function vector F. Such equations were first proposed by SIMMONDS and DANIELSON [247,248].

When rotations are taken as independent variables the rotational compatibility conditions $(5.24)_2$ or $(5.73)_{3,4}$ are identically satisfied. The force equilibrium equations $(5.57)_1$ can also be satisfied if we introduce the stress function vector $F = F^\alpha r_\alpha + F\bar{n}$ such that

$$\hat{N}^\beta = \epsilon^{\beta\alpha} F_{,\alpha} + P^\beta \quad , \qquad P^\beta = P^{\alpha\beta} r_\alpha + P^\beta \bar{n} \quad , \tag{5.87}$$

where P^β is a particular solution of (5.34). Now it follows from $(5.87)_1$ and $(5.47)_1$ that $S^{\alpha\beta}$, S and Q^β are prescribed functions of θ, F. It remains to satisfy the moment equilibrium equations $(5.57)_2$, the tangential compatibility conditions $(5.24)_1$ (since here u will not be regarded as an independent variable) and to eliminate u and B from the boundary conditions.

Let conservative surface and boundary loads be defined in terms of potentials $\Phi[\bar{r},\bar{n}(\theta)]$, $\psi[\bar{r},\bar{n}(\theta)]$ by

$$p = - \frac{\partial\Phi}{\partial\bar{r}} \quad , \qquad h = - \frac{\partial\Phi}{\partial\bar{n}} \quad , \qquad T = - \frac{\partial\psi}{\partial\bar{r}} \quad , \qquad H = - \frac{\partial\psi}{\partial\bar{n}} \quad . \tag{5.88}$$

Let us apply the Legendre transformation (3.36) only to the first part Σ_η of the strain energy density given in (5.77), which contains squares of $\eta_{\alpha\beta}$. Let us also introduce the tangential compatibility conditions $(5.24)_1$ into the functional (5.86) with the help of the Lagrange multiplier F. If (5.19) is also a priori satisfied then (5.86) can be written in the form

$$J_1 = \iint_{\mathcal{M}} \{ S^{\alpha\beta}\eta_{\alpha\beta} - \Sigma_S^C(S^{\alpha\beta}) + \Sigma_\rho(\rho_{\alpha\beta}) + \frac{1}{2} S\epsilon^{\alpha\beta}\eta_{\alpha\beta} + Q^\beta\bar{n}\cdot\eta_{\alpha\beta}r^\alpha -$$

$$- G^{\alpha\beta}[\rho_{\alpha\beta}- \rho_{\alpha\beta}(\theta)] + \epsilon^{\alpha\beta}(\epsilon_{\alpha|\beta}+ l_\beta \times r_\alpha)\cdot F + \Phi[\bar{r},\bar{n}(\theta)]\}dA +$$

(5.89)

$$+ \int_{\mathcal{C}_f} \varphi[\bar{r},\bar{n}(\theta)]ds - \int_{\mathcal{C}} (A\epsilon^{\alpha\beta}r_\alpha\cdot\eta_{\lambda\beta}r^\lambda + B^\beta\bar{n}\cdot\eta_{\lambda\beta}r^\lambda)ds -$$

$$- \int_{\mathcal{C}_u} [L\cdot(\bar{r} - \bar{r}^*) + M\cdot[\bar{n}(\theta) - \bar{n}(\theta^*)]\}ds - \sum_i K_i\cdot(\bar{r}_i- \bar{r}_i^*) ,$$

where A, B^β and L, M, K_i are corresponding Lagrange multipliers associated with the constraints (5.27) on $\mathcal{C}$ and with the geometric boundary and corner conditions (5.60) and (5.61).

The variational principle $\delta J_1 = 0$ allows us to find various stationarity conditions of J_1, among which are relations which identify the Lagrange multipliers A, B_ν, L, M, K_i to be

$$A = 0 , \quad B_\nu = 0 ,$$

$$L = P^\beta\nu_\beta + F' + (B_t\bar{n})', \quad M = G^{\alpha\beta}r_\alpha\nu_\beta - B_t\bar{a}_t ,$$

(5.90)

$$K_i = F_i + (B_t\bar{n})_i .$$

In order to eliminate the free field variables $\eta_{\alpha\beta}$, $G^{\alpha\beta}$, $\rho_{\alpha\beta}$, $S^{\alpha\beta}$, S, Q^β in $\mathcal{M}$, $\bar{r}$ on $\mathcal{C}_u$ and $\bar{r}_i$ at each $M_i \epsilon \mathcal{C}_u$ let us assume that the following stationarity conditions of J_1 are a priori satisfied

$$\eta_{\alpha\beta} = \frac{\partial\Sigma_S^C}{\partial S^{\alpha\beta}} , \quad G^{\alpha\beta} = \frac{\partial\Sigma_\rho}{\partial\rho_{\alpha\beta}} , \quad \rho_{\alpha\beta} = \rho_{\alpha\beta}(\theta), \quad \hat{N}^\beta = \epsilon^{\beta\alpha}F_{,\alpha} + P^\beta \text{ in } \mathcal{M},$$

(5.91)

$$\bar{r} = \bar{r}^* \text{ on } \mathcal{C}_u, \quad \bar{r}_i = \bar{r}_i^* \text{ on each } M_i \epsilon \mathcal{C}_u .$$

If now (5.90) and (5.91) are used then the functional (5.89) is reduced to

$$J_2 = \iint\limits_{\mathcal{M}} \{ (\epsilon^{\beta\alpha}F_{,\alpha} + P^\beta) \cdot \epsilon_\beta(F,\theta) - \Sigma_S^C(F,\theta) + \Sigma_\rho(\theta) +$$

$$+ \epsilon^{\alpha\beta}[\epsilon_{\alpha|\beta}(F,\theta) + l_\beta(\theta) \times r_\alpha(\theta)] \cdot F + P^\beta|_\beta \cdot \bar{r} + f[\bar{n}(\theta)]\}dA +$$

<div align="right">(5.92)</div>

$$+ \int\limits_{\mathcal{C}_f} \{- T \cdot \bar{r} + g[\bar{n}(\theta)]\}ds - \int\limits_{\mathcal{C}} B_t \bar{n}(\theta) \cdot \epsilon_\beta(F,\theta)t^\beta ds -$$

$$- \int\limits_{\mathcal{C}_u} [G^{\alpha\beta}(\theta)r_\alpha(\theta)\nu_\beta - B_t \bar{a}_t(F,\theta)] \cdot [\bar{n}(\theta) - \bar{n}(\theta^*)]ds ,$$

where $\Phi = P^\beta|_\beta \cdot \bar{r} + f[\bar{n}(\theta)]$ and $\Psi = -T \cdot \bar{r} + g[\bar{n}(\theta)]$ have been used.

Since $\bar{a}_{\alpha|\beta} = l_\beta \times r_\alpha + b_{\alpha\beta}\bar{n} + \epsilon_{\alpha|\beta}$ in $\mathcal{M}$, the second line of (5.92) can be transformed further to the form

$$\iint\limits_{\mathcal{M}} [- (\epsilon^{\beta\alpha}F_{,\alpha} + P^\beta) \cdot \bar{a}_\beta + f]dA +$$

<div align="right">(5.93)</div>

$$+ \int\limits_{\mathcal{C}_f} (P^\beta\nu_\beta \cdot \bar{r} - F \cdot \bar{r}')ds + \int\limits_{\mathcal{C}_u} (P^\beta\nu_\beta \cdot \bar{r}^* - F \cdot \bar{r}^{*'})ds .$$

It follows from (5.92) and (5.93) that on $\mathcal{C}_f$ we still have to eliminate $\bar{r}$ from the following line integral

$$\int\limits_{\mathcal{C}_f} (P^\beta\nu_\beta \cdot \bar{r} - F \cdot \bar{r}' - T \cdot \bar{r})ds =$$

<div align="right">(5.94)</div>

$$= \int\limits_{\mathcal{C}_f} (P^\beta\nu_\beta + F' - T) \cdot \bar{r} - \sum_j [F(s_{j+1}-0) \cdot \bar{r}_{j+1} - F(s_j+0) \cdot \bar{r}_j] .$$

It is easy to see that the values of $\bar{r}$ on each $M_j \in \mathcal{C}_f$ are not known, in general, and the out-of-integral terms in $(5.94)_2$ can not be evaluated only in terms of F and θ. However, there are two special cases of the boundary conditions when those terms are given. The first obvious case is when the boundary contour $\mathcal{C}$ has no corner points. In this case those terms do not appear at all. The second special case is when on $\mathcal{C}$ only (displacement) geometric boundary conditions are prescribed, i.e. $\mathcal{C} \equiv \mathcal{C}_u$, or $\mathcal{C}$ is divided by the corner points into an even number m of intervals, on which alternately only static $(5.58)_1$ or only geometric $\bar{r} = \bar{r}^*$ boundary conditions are prescribed. In the last case all the corner points belong simultaneously to $\mathcal{C}_f$ and to $\mathcal{C}_u$. Intervals $(M_j, M_{j+i}) \in \mathcal{C}_f$, $(M_i, M_{i+1}) \in \mathcal{C}_u$, where $j = 1,3,5,\ldots, m-1$, $i = j+1$. Since deformation of $\mathcal{C}$ is continuous and $\bar{r}_i = \bar{r}_i^*$ on each $M_i \in \mathcal{C}_u$ we indeed obtain $\bar{r}_j = \bar{r}_{i-1}^*$ for any $M_j \in \mathcal{C}_f$.

Let us assume that $\mathcal{C}$ is divided indeed into an even number of intervals on which alternately only respective static or geometric quantities are prescribed, as discussed above. Then in order to eliminate $\bar{r}$ from the line integral of $(5.94)_2$ the following functions are introduced on $\mathcal{C}_f$

$$G_j(s) = \int_{s_j}^{s} (T - P^\beta \nu_\beta)ds \ , \quad T - P^\beta \nu_\beta = (G_j + C_j)' \ , \tag{5.95}$$

where C_j are constant vectors which should ensure F to approach $\mathcal{C}_f$ continuously. Taking further into account that $G_j(s_j) \equiv 0$, $\bar{r}(s_j) = \bar{r}_{i-1}^* = \bar{r}_j^*$, $\bar{r}(s_{j+1}) = \bar{r}_i^* = \bar{r}_{j+1}^*$ we can transform $(5.94)_2$ into

$$\sum_{j=1,3,\ldots}^{m-1} \{ \int_{s_j}^{s_{j+1}} [(G_j+ C_j- F)\cdot\bar{r}']ds - [G_j(s_{j+1})\cdot\bar{r}_{j+1}^* + C_j(\bar{r}_{j+1}^*- \bar{r}_j^*)]\} \tag{5.96}$$

where $\bar{r}'$ on $\mathcal{C}_f$ is understood to be expressed in terms of F, θ by $(5.7)_2$, the inverse of $(5.78)_1$, $(5.47)_1$ and $(5.87)_1$. As a result, the functional J_2 in (5.92) can be transformed into the form

$$J_2(F,\theta,B_t,C_j) = \iint\limits_{\mathcal{M}} \{\Sigma_\rho(\theta) - \Sigma_S^C(F,\theta) - (\epsilon^{\beta\alpha}F_{,\alpha} + P^\beta)\cdot r_\beta(\theta) + f[\bar{n}(\theta)]\}dA +$$

$$+ \sum_{j=1,3,\dots}^{m-1} \left(\int\limits_{s_j}^{s_{j+1}} \{(G_j + C_j - F)\cdot\bar{r}'(F,\theta) + g[\bar{n}(\theta)]\}ds - \right.$$

(5.97)

$$\left. - [G_j(s_{j+1})\cdot\vec{r}_{j+1}^* + C_j(\vec{r}_{j+1}^* - \vec{r}_j^*)]\right) - \int\limits_{\mathcal{C}} B_t\bar{n}(\theta)\cdot\epsilon_\beta(F,\theta)t^\beta ds -$$

$$- \int\limits_{\mathcal{C}_u} \{F\cdot\vec{r}^{*\,'} - P^\beta\nu_\beta\cdot\vec{r}^* + [G^{\alpha\beta}(\theta)r_\alpha(\theta)\nu_\beta - B_t\bar{r}'(F,\theta)]\cdot[\bar{n}(\theta) - \bar{n}(\theta^*)]\} \, ds \, .$$

The variational principle $\delta J_2 = 0$ is equivalent to three tangential compatibility conditions $(5.24)_1$ in $\mathcal{M}$, three moment equilibrium equations $(5.57)_2$ in $\mathcal{M}$, three constraint conditions (5.27) in $\mathcal{M}$, three force static boundary conditions $G_j + C_j - F - B_t\bar{n} = 0$ on each interval $(M_j, M_{j+1}) \in \mathcal{C}_f$, $j=1,3,\dots,m-1$, two relations (5.63) on $\mathcal{C}_f$ (the first identifies B_t and the second is the static boundary condition for the couple), one constraint condition $\bar{n}\cdot\epsilon_\beta t^\beta = 0$ on $\mathcal{C}$ and three geometric boundary conditions $(5.60)_2$ on $\mathcal{C}_u$ for the rotations.

If we compare (5.97) with the analogous functional given by SIMMONDS and DANIELSON [248, f.(76)] we note that, apart of some unimportant constant terms and the extended potentials f and g which are included into (5.97), also the line integral over $\mathcal{C}$ and the last term in the line integral over $\mathcal{C}_u$ in (5.97) do not appear in the corresponding functional of [248]. Even if B_t is eliminated from (5.97) with the help of $(5.63)_1$, those two line integrals do not reduce themselves and have to be taken into account in the consistent non-linear theory of shells, which is expressed in terms of stress functions and finite rotations as independent variables.

Several functionals in terms of finite rotations were discussed also by ATLURI [14], who used the undeformed as well as the rotated basis as a reference basis. In the reduced forms of the functionals of [14] also the force static boundary

conditions were supposed to be a priori satisfied. This means that corresponding C_j should be constructed separately outside the variational problem. Term analogous to the last one in (5.97) is taken into account in [14], but the line integral over $\mathcal{C}$ of (5.97) still does not appear in the corresponding functional of [14], what makes the solution even more difficult. In the functional proposed recently by BAŞAR [24,25] the rotation vector has been defined as $\omega = n \times \bar{n} = \varepsilon^{\alpha\beta} n_\alpha a_\beta$, cf. [23]. So defined vector has different geometric meaning than the finite rotation vector used here and, therefore, the functional of [25] can not be compared with the functionals discussed here. Under $K - L$ contraints the functional of [25] becomes a particular case of the functional (4.8) of the displacement shell theory developed in [201,197].

In the literature on computerized FE analysis of flexible shells rotations are utilized explicitly and implicitly, exactly and approximately, on the level of an element and in the global matarices. As a result, it is not apparent how to compare the theoretical shell model discussed here with the numerical shell models. Let us only note that rotations were used in the numerical shell models proposed, among others, by RAMM [203], ARGYRIS et al. [12], PARISCH [181], HUGHES and LIU [92], SURANA [265], OLIVER and OÑATE [180], BERGAN and NYGARD [29], RECKE and WUNDERLICH [204] and RECKE [205] where further references are given.

6. Intrinsic shell equations

In some special problems of flexible shells, under particular types of boundary conditions, the basic set of non-linear shell equations may be expressed entirely in terms of two-dimensional strain and/or stress measures. Such intrinsic shell equations and their approximate versions for the geometrically non-linear bending theory of thin elastic shells were derived already by CHIEN [44] in terms of the strain measures. Alternative sets of intrinsic shell equations and/or alternative schemes of their approximation were proposed by MUSHTARI [152], ALUMÄE [5], KOITER [115], JOHN [101-103], WESTBROOK [294], AXELRAD [15,17] and VALID [283]. Intrinsic formulations of thin shell dynamics were discussed in [84,130,131,304]. DANIELSON [49] selected stress resultants and changes of curvatures as basic independent variables what allowed him to derive the refined set of intrinsic shell equations. Those equations were then modified slightly by KOITER an SIMMONDS [120] with the help of JOHN's [101] error estimates. Alternative formulations and special cases of the refined intrinsic shell equations were discussed by PIETRASZKIEWICZ [185,190], SIMMONDS [244], LIBAI and SIMMONDS [133] and KOITER [110].

The simplicity of the intrinsic shell equations is remarkable. Their solution leads directly to the determination of stress and strain distributions in the shell, without necessity to calculate displacements. However, displacements and/or rotations may be calculated, if necessary, by an additional integration of the kinematic relations (2.7), (2.10) or (5.19), (5.20).

6.1 Intrinsic bending shell equations

Let us note that the component form (3.31) of the mixed shell equations $(3.30)_1$ in the deformed basis $\bar{a}_\alpha$, $\bar{n}$ is already expressed entirely in terms of two-dimensional strain and stress measures. Corresponding four static boundary and three corner conditions $(3.30)_{2,3}$, when written in components along $\bar{v}$, $\bar{t}$, $\bar{n}$, are also expressed in terms of the strain and stress measures. Appropriate boundary conditions on $\mathcal{C}_u$ can also be expressed entirely in terms of the strain measures by assuming (5.34) and γ_{tt} to be given on $\mathcal{C}_u$. Therefore, the equilibrium equations (3.31) and the compatibility conditions (2.12) constitute the basic set of six non-linear equations with respect to arbitrarily chosen six components of strain and/or stress measures which are connected by the constitutive equations (3.34) and (3.35).

Let us now assume that the small strains in the shell caused by stretching and bending of its reference surface are of comparable order in the whole shell, i.e. $\gamma_{\alpha\beta} \sim h\varkappa_{\alpha\beta}$. Then within the error of the first approximation to the strain energy density $(3.32)_1$ the equilibrium equations (3.31) and the compatibility conditions (2.12) can be essentially reduced [185,193] to the form

$$\frac{Eh}{1-\nu^2} [(1-\nu)\gamma_\alpha^\beta|_\beta + \nu\gamma_\beta^\beta|_\alpha] + q_\alpha = O(Eh\,\frac{\eta\theta^2}{\lambda}) \ ,$$

$$\frac{Eh^3}{12(1-\nu^2)} \varkappa_\alpha^\alpha|_\beta + \frac{Eh}{1-\nu^2} (b_\beta^\alpha - \varkappa_\beta^\alpha)[(1-\nu)\gamma_\alpha^\beta + \nu\delta_\alpha^\beta\gamma_\lambda^\lambda] + q + k^\alpha|_\alpha = O(Eh^2\,\frac{\eta\theta^2}{\lambda^2}) \ ,$$
$$(6.1)$$

$$\varkappa_\alpha^\beta|_\beta - \varkappa_\beta^\beta|_\alpha = O(\frac{\eta\theta^2}{h\lambda}) \ ,$$

$$\gamma_\alpha^\beta|_\beta^\alpha - \gamma_\alpha^\alpha|_\beta^\beta - (b_\alpha^\beta\varkappa_\beta^\alpha - b_\alpha^\alpha\varkappa_\beta^\beta) + \frac{1}{2}(\varkappa_\alpha^\beta\varkappa_\beta^\alpha - \varkappa_\alpha^\alpha\varkappa_\beta^\beta) = O(\frac{\eta\theta^2}{\lambda^2}) \ .$$

Corresponding static boundary conditions on $\mathcal{C}_f$ reduce to [193]

$$\frac{Eh}{1-\nu^2} (\gamma_{\nu\nu} + \nu\gamma_{tt}) + O(Eh\eta\theta^2) = Q_\nu \ ,$$

$$\frac{Eh}{1-\nu^2} (1-\nu)\gamma_{\nu t} + O(Eh\eta\theta^2) = Q_t \ ,$$

$$\frac{Eh^3}{12(1-\nu^2)} \{\varkappa_{\nu\nu,\nu} + \nu\varkappa_{tt,\nu} + 2(1-\nu)\varkappa_{\nu t}' +$$
$$(6.2)$$

$$+(1-\nu)[\varkappa_t(\varkappa_{\nu\nu} - \varkappa_{tt}) + 2\varkappa_\nu\varkappa_{\nu t}]\} + O(\frac{Eh^2\eta\theta^2}{\lambda}) = Q + K_t' \ ,$$

$$\frac{Eh^3}{12(1-\nu^2)} (\varkappa_{\nu\nu} + \nu\varkappa_{tt}) + O(Eh^3\eta\theta^2) = K_\nu \ ,$$

while at each corner point $M_j \in \mathcal{C}_f$ we should assume

$$\frac{Eh^3}{12(1-\nu^2)} (1-\nu)(\varkappa_{\nu t})_j\bar{n}_j + O(Eh^2\eta\theta^2) = (K_t)_j\bar{n}_j \ . \qquad (6.3)$$

Corresponding deformational quantities (5.34) can also be reduced in accordance with the error already introduced into the reduced compatibility conditions

$(6.1)_{3,4}$. This gives us the deformational boundary conditions [193] on $\mathcal{C}_u$:

$$\varkappa_{tt} + O(\frac{\eta\theta^2}{h}) = k^*_{tt} , \qquad \varkappa_{\nu t} + O(\frac{\eta\theta^2}{h}) = k^*_{\nu t} ,$$

$$2\gamma'_{\nu t} - \gamma_{tt,\nu} + 2\varkappa_\nu\gamma_{\nu t} + \varkappa_t(\gamma_{\nu\nu} - \gamma_{tt}) + O(\frac{\eta\theta^3}{h}) = k^*_{nt} , \qquad (6.4)$$

$$\gamma_{tt} = \gamma^*_{tt} .$$

The resulting set of bending intrinsic shell relations (6.1) - (6.4) is very simple. Four field equations $(6.1)_{1,3}$ are linear while two remaining ones $(6.1)_{2,4}$ are quadratic in terms of $\gamma_{\alpha\beta}$ and $\varkappa_{\alpha\beta}$. All boundary conditions are linear in the strain measures.

6.2. Refined intrinsic shell equations

In many problems of flexible shells the small strains caused by membrane force resultants may be of essentially different order (higher or smaller by the factor θ^2) from those caused by the couple resultants. In those cases the reduced bending shell equations $(6.1)_{1,3}$ should be approximated with a greater accuracy, since within the accuracy indicated in $(6.1)_{1,3}$ they contain only terms of one kind: membrane strains or changes of curvatures, respectively.

The refinement of $(6.1)_{1,3}$ may be performed by selecting membrane stress resultants $N^{\alpha\beta}$ and changes of curvatures $\varkappa_{\alpha\beta}$ as the basic independent variables of the shell theory. The estimation procedure presented in detail in [49,120,185,193] leads then to the following refined intrinsic shell equations

$$N^\beta_{\alpha|\beta} + 2A(N^\lambda_\alpha N^\beta_\lambda)_{|\beta} - \frac{1}{2} A[(1-\nu)N^\lambda_\beta N^\beta_\lambda + \nu N^\lambda_\lambda N^\beta_\beta]|_\alpha -$$

$$- D\{(b^\lambda_\alpha - \varkappa^\lambda_\alpha)[(1-\nu)\varkappa^\beta_\lambda + \nu\delta^\beta_\lambda\varkappa^\varkappa_\varkappa]\}_{|\beta} - (b^\beta_\alpha - \varkappa^\beta_\alpha)(D\varkappa^\lambda_{\lambda|\beta} + k_\beta) +$$

$$+ 2A[(1+\nu)N^\beta_\alpha q_\beta - \nu N^\lambda_\lambda q_\alpha] + q_\alpha = O(Eh \frac{\eta\theta^4}{\lambda}) ,$$

$$D\varkappa^\alpha_{\alpha|\beta}\,^\beta + (b^\beta_\alpha - \varkappa^\beta_\alpha)N^\alpha_\beta + q + k^\alpha|_\alpha = O(Eh^2 \frac{\eta\theta^2}{\lambda^2}) , \qquad (6.5)$$

$$x^\beta_{\alpha|\beta} - x^\beta_{\beta|\alpha} - A(1+\nu)[(b^\lambda_\beta - x^\lambda_\beta)N^\beta_{\lambda|\alpha} + (b^\beta_\alpha - x^\beta_\alpha)N^\lambda_{\lambda|\beta}] +$$

$$+ A\nu(b^\beta_\beta - x^\beta_\beta)N^\lambda_{\lambda|\alpha} - 2A(1+\nu)(b^\beta_\alpha - x^\beta_\alpha)q_\beta = O(\frac{n\theta^4}{h\lambda}) ,$$

$$AN^\alpha_{\alpha|\beta} + (b^\beta_\alpha - \frac{1}{2}x^\beta_\alpha)x^\alpha_\beta - (b^\alpha_\alpha - \frac{1}{2}x^\alpha_\alpha)x^\beta_\beta + A(1+\nu)q^\beta_{|\beta} = O(\frac{n\theta^2}{\lambda^2}) ,$$

$$A = \frac{1}{Eh} , \qquad D = \frac{Eh^3}{12(1-\nu^2)} . \qquad\qquad (6.6)$$

The refined intrinsic shell equations (6.5) expressed in terms of $N^{\alpha\beta}$, $x_{\alpha\beta}$ as independent variables and with all the external surface forces q_α, q taken into account were given in [185]. Here we have additionally supplemented them with the external surface moments k_α . DANIELSON [49] derived (6.5) in terms of $- x_{\alpha\beta}$ and a modified stress resultant tensor $n^{\alpha\beta}$, with only q taken into account. KOITER and SIMMONDS [120] expressed (6.5) in terms of $n^{\alpha\beta}$, $- \hat{\rho}_{\alpha\beta}$ in the absence of surface forces, while q_α, q was taken into account in [190].

Associated with (6.5) boundary and corner conditions should also be refined. Note that only the tangential static boundary conditions $(6.2)_{1,2}$ and the tangential deformational boundary conditions $(6.4)_1$ need to be refined, since the other boundary conditions $(6.2)_{3,4}$, (6.3) and $(6.4)_{2,3}$ of the bending shell theory are accurate enough for the use with the intrinsic equations (6.5).

Let us multiply $(3.30)_2$ by $\bar{v}$ or $\bar{t}$, apply the transformation rules (3.9) to express $\bar{v}_\alpha$ or $\bar{t}_\alpha$ in terms of v_α or t_α , respectively, and use the constitutive equations $(3.34)_2$. Then within the error $O(Eh\eta\theta^4)$ the tangential static boundary conditions $(3.30)_2$ on $\mathcal{C}_f$ can be reduced to the consistently approximated form

$$[1 + A(N_{vv} - \nu N_{tt})]N_{vv} - D(\sigma_v - x_{vv})(x_{vv} + \nu x_{tt}) + 2D(1-\nu)(\tau_t + x_{vt})x_{vt} =$$

$$= Q_v + (\tau_t + x_{vt})K_t + O(Eh\eta\theta^4) , \qquad\qquad (6.7)$$

$$[1 + A(N_{tt} - \nu N_{vv})]N_{vt} + 2A(1+\nu)N_{vv}N_{vt} + D(\tau_t + x_{vt})(x_{vv} + \nu x_{tt}) -$$

$$- 2D(1-\nu)(\sigma_t - x_{tt})x_{vt} = Q_t - (\sigma_t - x_{tt})K_t + O(Eh\eta\theta^4) .$$

Corresponding deformational boundary conditions on $\mathcal{C}_u$ can be constructed by the consistent reduction, to within the error $O(\frac{\eta\theta^4}{h})$, of the parameters k_{tt} and k_{vt} given in $(5.34)_{1,2}$ with the subsequent elimination of $\gamma_{\alpha\beta}$ with the help of the constitutive equations $(3.35)_1$. As a result, we obtain the following consistently approximated deformational boundary conditions

$$\varkappa_{tt} + A(\sigma_t - \varkappa_{tt})(N_{tt} - \nu N_{vv}) = k^*_{tt} + O(\frac{\eta\theta^4}{h}) \ ,$$

$$\varkappa_{vt} + 2A(1+\nu)(\sigma_t - \varkappa_{tt})N_{vt} - A(\tau_t + \varkappa_{vt})(N_{vv} - \nu N_{tt}) = k^*_{vt} + O(\frac{\eta\theta^4}{h}) \ ,$$

$$2A(1+\nu)N'_{vt} - A(N_{tt,v} - \nu N_{vv,v}) + 2A(1+\nu)\varkappa_{vt}N_{vt} + \tag{6.8}$$

$$+ A(1+\nu)\varkappa_t(N_{vv} - N_{tt}) = k^*_{nt} + O(\frac{\eta\theta^3}{h}) \ ,$$

$$A(N_{tt} - \nu N_{vv}) = \gamma^*_{tt} \ .$$

The static boundary conditions (6.7), $(6.2)_{3,4}$ are equivalent to those given by DANIELSON [49] in terms of $n^{\alpha\beta}$, $-\varkappa_{\alpha\beta}$. The refined form (6.8) of deformational boundary conditions has not been discussed in the literature.

6.3. Work-conjugate static boundary conditions

The consistently simplified static and deformational boundary conditions given in sections 6.1 and 6.2 are not work-conjugate to each other, since the static parameters in the line integral (3.28) work on virtual displacements and not on variations of the deformational parameters k_{tt}, k_{vt}, k_{nt}, γ_{tt} . In order to derive work-conjugate boundary conditions the line integral of (3.28) should be transformed as it was suggested in [192,193].

According to $(2.17)_1$ and $(5.30)_2$, $\bar{a}_t = t + u' = \bar{a}_t R_t t$. Taking the variation of this expression with the help of (4.27) and using the identity $\delta u' = (\delta u)'$ we obtain

$$(\delta\bar{u})' = \frac{1}{\bar{a}^2} \bar{a}_t \delta\gamma_{tt} + \delta\omega_t \times \bar{a}_t \ , \tag{6.9}$$

where by $\delta\bar{u}$ we understand the variation of displacement field on $\mathcal{C}$, which is referred then to the deformed basis $\bar{v}, \bar{t}, \bar{n}$, i.e. the virtual displacement field appearing in (3.28)

Let c be an arbitrary constant vector and $\bar{c} = R_t c$. Then $\delta\bar{c} = \delta\omega_t \times c$ and $\bar{c}' = 1_t \times c$, according to (4.27) and (5.31). Since again $(\delta\bar{c})' = \delta(\bar{c}')$ this leads to

$$(\delta\omega_t)' = \delta 1_t - \delta\omega_t \times 1_t . \tag{6.10}$$

Using (5.31) and (5.33)$_2$ the relation (6.10) can also be presented in the alternative form

$$(\delta\omega_t)' = - \delta k_{tt}\bar{v} + \delta k_{vt}\bar{t} - \delta k_{nt}\bar{n} .$$

Let A and B(0) be the vectors of the total force and the total couple with respect to the origin $O \in \mathcal{C}$ of all the internal force and couple resultants acting on a part of the deformed boundary $\mathcal{C}$. With the help of transformation rule (3.9)$_1$ these vectors are defined by

$$A = A_0 + \int_{s_0}^{s} P ds , \qquad P = N^\beta v_\beta - (H\bar{n})' ,$$

$$\tag{6.12}$$

$$B(0) = B_0(0) + \int_{s_0}^{s} (\bar{r} \times P + G\bar{t}) ds ,$$

where N^β, H and G are given in (3.29) and A_0, $B_0(0)$ are initial values of A, B(0) . The total couple vector $B = B(\bar{M})$ with respect to a current point $\bar{M}$ of $\mathcal{C}$ is given by

$$B = B(0) - \bar{r} \times A . \tag{6.13}$$

Differentiating A and B along $\mathcal{C}$ we obtain

$$A' = P , \quad B' = G\bar{t} - \bar{a}_t \times A .$$ (6.14)

Differential relations (6.9), (6.11) and (6.14) can be used to transform the boundary terms in (3.28). Indeed, introducing A' for P into (3.28) and integrating by parts, then again introducing B' for $G\bar{t} - \bar{a}_t \times A$ and again integrating by parts we obtain

$$\int_{\mathcal{C}_f} (P \cdot \delta\bar{u} + G\bar{t} \cdot \delta\omega_t) ds - \sum_j H_j \bar{n}_j \cdot \delta\bar{u}_j =$$ (6.15)

$$= - \int_{\mathcal{C}_f} [B \cdot (\delta\omega_t)' + \frac{1}{\bar{a}_t^2} A \cdot \bar{a}_t \delta\gamma_{tt}] ds - \sum_j [(H_j \bar{n}_j + A_j) \cdot \delta\bar{u}_j + B_j \cdot \delta\omega_{tj}] .$$

Exactly the same transformations hold for the analogous external force and couple resultant vectors, only in this case T, H^*, G^* and H_j^* appear in place of $N^\beta \nu_\beta$, H, G and H_j , respectively, in analogous definitions of P^*, A^*, B^*, A_j^* and B_j^* . As a result, with the help of (6.11), (6.15) and an analogous transformed integral for the starred quantities the boundary terms of (3.28) can be transformed into

$$\int_{\mathcal{C}_f} [(B-B^*) \cdot \bar{\nu}\delta k_{tt} - (B-B^*) \cdot \bar{t}\delta k_{\nu t} + (B-B^*) \cdot \bar{n}\delta k_{nt} - \frac{1}{\bar{a}_t}(A-A^*) \cdot \bar{t}\delta\gamma_{tt}] ds -$$ (6.16)

$$- \sum_j \{[(H_j - H_j^*)\bar{n}_j + A_j - A_j^*] \cdot \delta\bar{u}_j + (B_j - B_j^*) \cdot \delta\omega_{tj}\} .$$

It is apparent from (6.16) that on $\mathcal{C}_f$ components of $B - B^*$ in the basis $\bar{\nu}$, $\bar{t}$, $\bar{n}$ and the component $-\bar{a}_t^{-1}(A - A^*) \cdot \bar{t}$ work on variations of the deformational parameters. Therefore, static boundary conditions which are work-conjugate to the deformational ones have the form

$$B = B^* \ , \qquad \frac{1}{\bar{a}_t} A \cdot \bar{t} = \frac{1}{\bar{a}_t} A^* \cdot \bar{t} \quad \text{on} \ \mathcal{C}_f \ . \tag{6.17}$$

It follows from (6.16) that terms associated with the virtual work at the corners $M_j \ \epsilon \ \mathcal{C}_f$ are not expressed in the intrinsic form, since the static parameters work there on $\delta \bar{u}$ and $\delta \omega_t$, respectively, but not on variations of deformational quantities. Therefore, in oder a shell problem be solvable in the intrinsic way, entirely in terms of strain and/or stress measures, those out-of-integral terms should identically vanish. It is easy to note that those terms vanish identically in the case of the smooth boundary contour (i. e. without corners) or when only geometric boundary conditions are assumed on the entire $\mathcal{C}$. Another special case is when $\mathcal{C}$ is divided by corners into an even number of intervals, on which alternately only static or only geometric boundary conditions are prescribed. In such a case all the corners belong simultaneously to $\mathcal{C}_f$ and to $\mathcal{C}_u$ and, therefore, $\delta \bar{u}$ and $\delta \omega_t$ vanish identically at each corner $M \ \epsilon \ \mathcal{C}$.

When the work-conjugate static boundary conditions (6.17) are used in conjunction with the bending shell equations (6.1) or with the refined ones (6.5), all the vectors A , B , A^* and B^* should be calculated from the consistently reduced components of P , G , P^* and G^* given in (6.2) and (6.7), respectively.

6.4. Alternative form of the refined intrinsic shell equations

An alternative set of intrinsic shell equations was derived in chapter 5. Indeed, six equilibrium equations (5.51), (5.62) and six compatibility conditions (5.73) are expressed entirely in terms of the stress measures $S^{\alpha\beta}$, $G^{\alpha\beta}$, Q^β, S and the strain measures $\eta_{\alpha\beta}$, $\rho_{\alpha\beta}$, k_β, ρ .

When strains are small everywhere, the equilibrium equations (5.62), (5.51) can be reduced within the error of the first approximation theory to the form

$$G^{\alpha\beta}\big|_\beta - Q^\alpha + \hat{h}^\alpha = O(Eh^2 \frac{\eta\theta^2}{\lambda}) \ ,$$

$$S - \epsilon_{\alpha\lambda}\eta_\beta^\alpha S^{\lambda\beta} - \epsilon_{\alpha\lambda}G^{\alpha\beta}(b_\beta^\lambda - \rho_\beta^\lambda) = O(Eh\eta\theta^4) \ ,$$

$$\tag{6.18}$$

$$S^{\alpha\beta}\big|_\beta - \epsilon^{\alpha\lambda}S_\lambda^\beta k_\beta + \frac{1}{2}\epsilon^{\alpha\beta}S_{,\beta} - Q^\beta(b_\beta^\alpha - \rho_\beta^\alpha) + \hat{p}^\alpha = O(Eh \frac{\eta\theta^2}{\lambda}) \ ,$$

$$S^{\alpha\beta}(b_{\alpha\beta} - \rho_{\alpha\beta}) + Q^{\beta}|_{\beta} + \hat{p} = O(Eh^2 \frac{\eta\theta^2}{\lambda^2}) \ .$$

Similarly, the compatibility conditions (5.73) can be reduced into

$$\varepsilon^{\alpha\beta} \eta_{\lambda\alpha}|_{\beta} + k_{\lambda} = O(\frac{\eta\theta^2}{\lambda}) \ ,$$

$$\rho - \varepsilon^{\alpha\beta} \eta_{\alpha\varkappa}(b^{\varkappa}_{\beta} - \rho^{\varkappa}_{\beta}) = O(\frac{\eta\theta^4}{h}) \ ,$$

$$\varepsilon^{\alpha\beta} \varepsilon^{\lambda\varkappa} \rho_{\lambda\alpha}|_{\beta} + \frac{1}{2} \varepsilon^{\varkappa\beta} \rho_{,\beta} - \varepsilon^{\alpha\beta} k_{\alpha}(b^{\varkappa}_{\beta} - \rho^{\varkappa}_{\beta}) = O(\frac{\eta\theta^4}{h\lambda}) \ ,$$

$$\varepsilon^{\alpha\beta} \varepsilon^{\lambda\varkappa}(b_{\lambda\alpha} - \frac{1}{2} \rho_{\lambda\alpha})\rho_{\varkappa\beta} + \varepsilon^{\alpha\beta} k_{\alpha}|_{\beta} = O(\frac{\eta\theta^2}{\lambda^2}) \ .$$

$$(6.19)$$

It should be noted that the static-geometric analogy formulated in section 5.5 holds also between the reduced sets of equations (6.18), (6.19).

Let us solve $(6.18)_{1,2}$ for Q^{α}, S and $(6.19)_{1,2}$ for k_{α}, ρ , respectively, and introduce them into the remaining equations (6.18), (6.19), what leads to

$$S^{\alpha\beta}|_{\beta} + \varepsilon^{\alpha\lambda} \varepsilon^{\varkappa\rho} S^{\beta}_{\lambda} \eta_{\beta\varkappa}|_{\rho} + \frac{1}{2}\varepsilon^{\alpha\beta} \varepsilon_{\varkappa\rho}[\eta^{\varkappa}_{\lambda} S^{\lambda\rho} + G^{\varkappa\lambda}(b^{\rho}_{\lambda} - \rho^{\rho}_{\lambda})]|_{\beta} -$$

$$- (G^{\beta\lambda}|_{\lambda} + \hat{h}^{\beta})(b^{\alpha}_{\beta} - \rho^{\alpha}_{\beta}) + \hat{p}^{\alpha} = O(Eh \frac{\eta\theta^4}{\lambda}) \ , \qquad (6.20)$$

$$S^{\alpha\beta}(b_{\alpha\beta} - \rho_{\alpha\beta}) + G^{\alpha\beta}|_{\alpha\beta} + \hat{h}^{\beta}|_{\beta} + \hat{p} = O(Eh^2 \frac{\eta\theta^2}{\lambda^2}) \ ,$$

$$\varepsilon^{\alpha\beta} \varepsilon^{\lambda\varkappa} \rho_{\lambda\alpha}|_{\beta} + \frac{1}{2} \varepsilon^{\varkappa\beta} \varepsilon^{\alpha\lambda}[\eta_{\alpha\rho}(b^{\rho}_{\lambda} - \rho^{\rho}_{\lambda})]|_{\beta} +$$

$$+ \varepsilon^{\alpha\beta} \varepsilon^{\lambda\rho} \eta_{\alpha\lambda}|_{\rho}(b^{\varkappa}_{\beta} - \rho^{\varkappa}_{\beta}) = O(\frac{\eta\theta^4}{h\lambda}) \ , \qquad (6.21)$$

$$\varepsilon^{\alpha\beta} \varepsilon^{\lambda\varkappa}[(b_{\lambda\alpha} - \frac{1}{2} \rho_{\lambda\alpha})\rho_{\varkappa\beta} - \eta_{\alpha\lambda}|_{\varkappa\beta}] = O(\frac{\eta\theta^2}{\lambda^2}) \ .$$

Introducing the transformations (5.74) into the left-hand sides of (6.21) and changing signs we obtain

$$\tilde{\rho}^{\alpha\beta}\big|_{\beta} + \frac{1}{2}\epsilon^{\alpha\lambda}\epsilon^{\varkappa\rho}\tilde{\rho}^{\beta}_{\lambda}\eta_{\beta\varkappa}\big|_{\rho} + \frac{1}{2}\epsilon^{\alpha\beta}\epsilon_{\varkappa\rho}[\frac{1}{2}\eta^{\varkappa}_{\lambda}\tilde{\rho}^{\lambda\rho} +$$

$$+ \tilde{\eta}^{\varkappa\lambda}(b^{\rho}_{\lambda}- \frac{1}{2}\rho^{\rho}_{\lambda})]\big|_{\beta} - \tilde{\eta}^{\beta\lambda}\big|_{\lambda}(b^{\alpha}_{\beta}- \frac{1}{2}\rho^{\alpha}_{\beta}) = 0 \ , \tag{6.22}$$

$$\tilde{\rho}^{\alpha\beta}(b_{\alpha\beta}- \frac{1}{2}\rho_{\alpha\beta}) + \tilde{\eta}^{\alpha\beta}\big|_{\alpha\beta} = 0 \ .$$

When (6.22) is compared with the homogeneous equations (6.20) it is seen that the static–geometric analogy still holds between the reduced equations (6.20) and (6.21), what was not the case for the exact sets 3+3 of the transformed intrinsic shell equations (cf. section 5.5).

The measures $\eta_{\alpha\beta}$ and $G^{\alpha\beta}$ can still be eliminated from (6.20) and (6.21) with the help of constitutive equations

$$\eta_{\alpha\beta} = A[(1+\nu)S_{\alpha\beta}- \nu a_{\alpha\beta}S^{\lambda}_{\lambda}] + O(\eta\theta^2) \ ,$$

$$\tag{6.23}$$

$$G^{\alpha\beta} = D[(1-\nu)\rho^{\alpha\beta}- \nu a^{\alpha\beta}\rho^{\lambda}_{\lambda}] + O(Eh^2\eta\theta^2) \ ,$$

what after transformations gives us the following alternative form of the refined intrinsic shell equations

$$S^{\beta}_{\alpha}\big|_{\beta} + A[(1+\nu)S^{\lambda}_{\alpha}- \nu\delta^{\lambda}_{\alpha}S^{\varkappa}_{\varkappa}]\big|_{\beta}S^{\beta}_{\lambda} - \frac{1}{2}A[(1+\nu)S^{\beta}_{\lambda}S^{\lambda}_{\beta}- \nu S^{\lambda}_{\lambda}S^{\beta}_{\beta}]\big|_{\alpha} -$$

$$- \frac{1}{2}D(1-\nu)(b^{\lambda}_{\alpha}\rho^{\beta}_{\lambda}- b^{\beta}_{\lambda}\rho^{\lambda}_{\alpha})\big|_{\beta} - D(b^{\beta}_{\alpha}- \rho^{\beta}_{\alpha})\rho^{\lambda}_{\lambda}\big|_{\beta} +$$

$$+ \hat{p}_{\alpha} - (b^{\beta}_{\alpha}- \rho^{\beta}_{\alpha})\hat{h}_{\beta} = O(Eh\frac{\eta\theta^4}{\lambda}) \ ,$$

$$D\rho^{\alpha}_{\alpha}\big|^{\beta}_{\beta} + (b^{\beta}_{\alpha}-\rho^{\beta}_{\alpha})S^{\alpha}_{\beta} + \hat{p} + \hat{h}^{\alpha}\big|_{\alpha} = O(Eh^2\frac{\eta\theta^2}{\lambda^2}) \ , \tag{6.24}$$

$$\rho^{\beta}_{\alpha}\big|_{\beta} - \rho^{\beta}_{\beta}\big|_{\alpha} + \frac{1}{2}A(1+\nu)[(b^{\lambda}_{\alpha}- \rho^{\lambda}_{\alpha})S^{\beta}_{\lambda} - (b^{\beta}_{\lambda}- \rho^{\beta}_{\lambda})S^{\lambda}_{\alpha}]\big|_{\beta} -$$

$$- A(b^{\beta}_{\alpha}- \rho^{\beta}_{\alpha})S^{\lambda}_{\lambda}\big|_{\beta} - A(1+\nu)(b^{\beta}_{\alpha}- \rho^{\beta}_{\alpha})\hat{p}_{\beta} = O(\frac{\eta\theta^4}{h\lambda}) \ ,$$

$$AS^\alpha_\alpha|^\beta_\beta + (b^\beta_\alpha - \frac{1}{2} \rho^\beta_\alpha)\rho^\alpha_\beta - (b^\alpha_\alpha - \frac{1}{2} \rho^\alpha_\alpha)\rho^\beta_\beta + A(1+\nu)\hat{p}^\alpha|_\alpha = O(\frac{\eta\theta^2}{\lambda^2}) .$$

If we apply the identity [49,185]

$$(S^\lambda_\alpha S^\beta_\lambda - S^\beta_\alpha S^\lambda_\lambda)|_\beta = \frac{1}{2} (S^\lambda_\beta S^\beta_\lambda - S^\beta_\beta S^\lambda_\lambda)|_\alpha \tag{6.25}$$

the first two of the equilibrium equations $(6.24)_1$ can be put into another equivalent forms

$$S^\beta_\alpha|_\beta + A[(1-\nu)S^\lambda_\alpha + \nu\delta^\lambda_\alpha S^\varkappa_\varkappa]|_\beta S^\beta_\lambda - \frac{1}{2} A[(1-\nu)S^\lambda_\beta S^\beta_\lambda + \nu S^\lambda_\lambda S^\beta_\beta]|_\alpha +$$

$$+ \frac{1}{2} D(1-\nu)(b^\beta_\lambda \rho^\lambda_\alpha - b^\lambda_\alpha \rho^\beta_\lambda)|_\beta - D(b^\beta_\alpha - \rho^\beta_\alpha)\rho^\lambda_\lambda|_\beta + \tag{6.26}$$

$$+ 2A\nu(S^\beta_\alpha \hat{p}_\beta - S^\beta_\beta \hat{p}_\alpha) + \hat{p}_\alpha - (b^\beta_\alpha - \rho^\beta_\alpha)\hat{h}_\beta = O(Eh \frac{\eta\theta^4}{\lambda}) ,$$

$$S^\beta_\alpha|_\beta + AS^\lambda_\alpha|_\beta S^\beta_\lambda - \frac{1}{2} A(S^\beta_\lambda S^\lambda_\beta)|_\alpha + \frac{1}{2} D(1-\nu)(b^\beta_\lambda \rho^\lambda_\alpha - b^\lambda_\alpha \rho^\beta_\lambda)|_\beta -$$

$$- D(b^\beta_\alpha - \rho^\beta_\alpha)\rho^\lambda_\lambda|_\beta + A\nu(S^\beta_\alpha \hat{p}_\beta - S^\beta_\beta \hat{p}_\alpha) + \hat{p}_\alpha - (b^\beta_\alpha - \rho^\beta_\alpha)\hat{h}_\beta = O(Eh \frac{\eta\theta^4}{\lambda}) . \tag{6.27}$$

The refined intrinsic shell equations (6.24) are fully equivalent to those given by (6.5). This can be shown directly if we take into account the transformation rules (5.11) and (5.40) written here in the appropriately approximated forms

$$\rho^\beta_\alpha = \varkappa^\beta_\alpha + \frac{1}{2} A(1+\nu)[(b^\lambda_\alpha - \varkappa^\lambda_\alpha)N^\beta_\lambda + (b^\beta_\lambda - \varkappa^\beta_\lambda)N^\lambda_\alpha] -$$

$$- A\nu(b^\beta_\alpha - \varkappa^\beta_\alpha)N^\lambda_\lambda + O(\frac{\eta\theta^4}{h}) = \varkappa^\beta_\alpha + O(\frac{\eta\theta^2}{h}) ,$$

$$\tag{6.28}$$

$$S^\beta_\alpha = N^\beta_\alpha + A[(1+\nu)N^\lambda_\alpha - \nu\delta^\lambda_\alpha N^\varkappa_\varkappa]N^\beta_\lambda - \frac{1}{2} D(1-\nu)(b^\lambda_\alpha \varkappa^\beta_\lambda + b^\beta_\lambda \varkappa^\lambda_\alpha) -$$

$$- D\nu(b^\beta_\alpha - \varkappa^\beta_\alpha)\varkappa^\lambda_\lambda + D(1-\nu)\varkappa^\lambda_\alpha \varkappa^\beta_\lambda + O(Eh\eta\theta^4) = N^\beta_\alpha + O(Eh\eta\theta^2) .$$

In the comparison of (6.24) and (6.5) one should also take into account that the tangential equilibrium equations $(6.24)_1$ are derived here from the components (with the subsequently lowered index α) of the vector equations $(5.49)_1$ in the rotated basis r_α , while the corresponding equations $(6.5)_1$ have been derived from the components (again with the subsequently lowered index α) of the vector equations $(3.30)_1$ in the deformed basis $\bar{a}_\alpha$. Therefore, we should also take into account the following transformation of the bases

$$r_\alpha = \sqrt{\frac{a}{\bar{a}}}\, \varepsilon^{\beta\rho}\varepsilon_{\alpha\lambda}(\delta^\lambda_\rho + \eta^\lambda_\rho)\bar{a}_\beta = [\delta^\beta_\alpha - \gamma^\beta_\alpha + O(\eta\theta^2)]\bar{a}_\beta$$

$$= \{\delta^\beta_\alpha - A[(1+\nu)N^\beta_\alpha - \nu\delta^\beta_\alpha N^\lambda_\lambda] + O(\eta\theta^2)\}\bar{a}_\beta \ .$$

$$(6.29)$$

If now (6.28) are introduced into (6.26) and the effect of change of the basis is taken into account according to (6.29) then, within the indicated accuracy, the equations (6.26) can be transformed into the equations $(6.5)_1$. Applying the same arguments also the remaining equations of (6.24) can be transformed into the corresponding equations of (6.5). Therefore, the sets of the refined intrinsic shell equations (6.24) and (6.5) are fully equivalent indeed.

The corresponding set of refined static boundary conditions in terms of $S^{\alpha\beta}$, $\rho_{\alpha\beta}$ can be derived from (6.7) and $(6.2)_{3,4}$ if we apply there the reversed transformation rules (6.28). Then after appropriate estimates and transformations we obtain on $\mathcal{C}_f$

$$S_{\nu\nu} - A(1+\nu)S^2_{\nu t} + D(1-\nu)(\tau_t + \rho_{\nu t})\rho_{\nu t} = Q_\nu + (\tau_t + \rho_{\nu t})K_t + O(Eh\eta\theta^4) \ ,$$

$$S_{\nu t} + A(1+\nu)S_{\nu\nu}S_{\nu t} + \frac{1}{2}D(1-\nu)(\sigma_\nu - \rho_{\nu\nu})\rho_{\nu t} -$$

$$- \frac{3}{2}D(1-\nu)(\sigma_t - \rho_{tt})\rho_{\nu t} + \frac{1}{2}D(1-\nu)(\tau_t + \rho_{\nu t})\rho_{\nu\nu} -$$

$$(6.30)$$

$$- \frac{1}{2}D(1-\nu)(\tau_t + \rho_{\nu t})\rho_{tt} = Q_t - (\sigma_t - \rho_{tt})K_t + O(Eh\eta\theta^4) \ ,$$

$$D\{\rho_{\nu\nu,\nu} + \nu\rho_{tt,\nu} + 2(1-\nu)\rho'_{\nu t} + (1-\nu)[\varkappa_t(\rho_{\nu\nu} - \rho_{tt}) + 2\varkappa_\nu\rho_{\nu t}]\} =$$

$$= Q + K'_t + O(Eh^2\frac{\eta\theta^2}{\lambda}) \ ,$$

$$D(\rho_{\nu\nu} + \nu\rho_{tt}) = K_{\nu} + O(Eh^2\eta\theta^2) \ ,$$

while at each corner point $M_j \in \mathcal{C}_f$ we have

$$D(1-\nu)(\rho_{\nu t})_j \bar{\mathbf{n}} = (K_t)_j \bar{\mathbf{n}}_j + O(Eh^2\eta\theta^2) \ . \tag{6.31}$$

Similarly, the corresponding set of refined deformational boundary conditions in terms of $S^{\alpha\beta}$, $\rho_{\alpha\beta}$ can be derived from (6.8) again by applying the reversed transformation rules (6.28). Then after appropriate estimates and transformations we obtain on $\mathcal{C}_u$

$$\rho_{tt} + A(1+\nu)(\tau_t + \rho_{\nu t})S_{\nu t} - A(\sigma_t - \rho_{tt})(S_{tt} - \nu S_{\nu\nu}) = k_{nt}^* + O(\frac{\eta\theta^4}{h}) \ ,$$

$$\rho_{\nu t} - \frac{1}{2}A(1+\nu)(\sigma_\nu - \rho_{\nu\nu})S_{\nu\nu} - \frac{1}{2}A(1+\nu)(\sigma_t - \rho_{tt})S_{\nu t} +$$

$$+ \frac{1}{2}A(1-\nu)(\tau_t + \rho_{\nu t})(S_{\nu\nu} + S_{tt}) = k_{\nu t}^* + O(\frac{\eta\theta^4}{h}) \ ,$$

$$\tag{6.32}$$

$$2A(1+\nu)S_{\nu t}' - A(S_{tt,\nu} - \nu S_{\nu\nu,\nu}) + 2A(1+\nu)\rho_{\nu t}S_{\nu t} +$$

$$+ A(1+\nu)\varkappa_t(N_{\nu\nu} - N_{tt}) = k_{nt}^* + O(\frac{\eta\theta^3}{h}) \ ,$$

$$A(S_{tt} - \nu S_{\nu\nu}) = \gamma_{tt}^* + O(\eta\theta^2) \ .$$

It should be noted that within the indicated error the homogeneous equations (6.24) may be shown to be equivalent to the ones proposed by KOITER and SIMMONDS [120]. In particular, when linearized, both sets of equations reduce to those of the "best" linear theory of thin shells according to [37]. However, a) our equations (6.24) are expressed in terms of the measures $S^{\alpha\beta}$, $\rho_{\alpha\beta}$ which appear naturally in the non-linear theory of shells (cf. chapter 5) while the corresponding equations of [120] are expressed in terms of some modified measures for which no exact Eulerian counterparts can be defined (cf. discussion in section 3.4); b) our equations (6.24) take into account all the surface loads $\hat{p}_\alpha \sim Eh\eta/\lambda$, $\hat{p} \sim Eh^2\eta/\lambda^2$, $\hat{h}_\alpha \sim Eh^2\eta/\lambda$, while those of [120] are given for the case of

zero surface loads (the loads $\bar{p}^\alpha$ and $\bar{p}$ have been included in [190,119]); c) our equations (6.24) follow from the set of 3+3 reduced shell equations (6.20), (6.21) which obey the static-geometric analogy in the non-linear range of deformation, while such an analogy can not be established between the initial relations of [120]; d) our equations (6.24) are supplemented by appropriately simplified static and deformational boundary conditions, while no such boundary conditions are given in [120].

6.5. Some special cases of intrinsic shell equations

As it was noted in the Introduction, already CHIEN [44] proposed a formal classification of approximate versions of his intrinsic equations under the assumption of a slowly varying geometry and slowly varying strain states of plates and shells. MUSHTARI [152] applied less formal qualitative analysis and constructed approximate versions of intrinsic equations for small and medium bending of shells and plates. In [152] several sets of intrinsic equations of the boundary layer type were also given. ALUMÄE [5] introduced the notion of wave length of deformation patterns and discussed 12 cases of intrinsic equations for the buckling analysis of shells which are shallow or almost shallow relative to deformation patterns. The solution of the most complete set of such equations was then reduced to the solution of two equations expressed in terms of stress and deformation functions F, W . Similar assumptions were applied independently by LIBAI [130] and KOITER [115] to derive the equations for quasi-shallow shells. However, no corresponding intrinsic boundary conditions were discussed in the papers referred to above.

The derivation of refined equations [49] provided new possibilities in the proper formulation of intrinsic equations for various types of shell problems. As a starting point for further discussion, three different but equivalent versions of the refined shell equations may be used: the one proposed by KOITER and SIMMONDS [120] and supplemented by the surface forces in [190], which is expressed in terms of some modified stress and strain measures, the one derived in terms of $N^{\alpha\beta}$, $\varkappa_{\alpha\beta}$ by the author [185,193] and summarized here as eqs. (6.5) – (6.8), as well as the one derived in terms of $S^{\alpha\beta}$, $\rho_{\alpha\beta}$ in this report, eqs. (6.24), (6.30) – (6.32). Referring to discussion after (6.32), the third version, as the most complete, seems to be preferable.

Let us look more carefully into the structure of (6.24). Let $\eta_{\alpha\beta} \sim \eta$ and

$h\rho_{\alpha\beta} \sim h\rho$ be the maximum extensional and bending strains, respectively. Let also L_η and L_ρ be the wave lengths of deformation patterns associated with the extensional and bending strains, such that $\eta^\beta_{\alpha}|_\gamma \sim \eta/L_\eta$ and $\rho^\beta_{\alpha}|_\gamma \sim \rho/L_\rho$, respectively. Then dividing $(6.24)_{1,2}$ by E and multiplying $(6.24)_{3,4}$ by h^2, we obtain the following order estimates for magnitudes of individual terms in the refined intrinsic shell equations (6.24):

$(6.24)_1$: $\quad \dfrac{h}{L_\eta}\cdot\eta$, $\dfrac{h}{L_\eta}\cdot\eta^2$, $\dfrac{h}{R}\dfrac{h}{1}\cdot h\rho$, $\dfrac{h}{R}\dfrac{h}{L_\rho}\cdot h\rho$, $\dfrac{h}{L_\rho}\cdot(h\rho)^2$,

$(6.24)_2$: $\quad (\dfrac{h}{L_\rho})^2\cdot h\rho$, $\dfrac{h}{R}\cdot\eta$, $\eta\cdot h\rho$,

(6.33)

$(6.24)_3$: $\quad \dfrac{h}{L_\rho}\cdot h\rho$, $\dfrac{h}{R}\dfrac{h}{1}\cdot\eta$, $\dfrac{h}{R}\dfrac{h}{L_\eta}\cdot\eta$, $\dfrac{h}{L_\eta}\cdot\eta\cdot h\rho$, $\dfrac{h}{L_\eta}\cdot\eta\cdot h\rho$

$(6.24)_4$: $\quad (\dfrac{h}{L_\eta})^2\cdot\eta$, $\dfrac{h}{R}\cdot h\rho$, $(h\rho)^2$.

Similar estimates can also be given for terms appearing in the static (6.30), (6.31) and deformational (6.32) boundary conditions.

Various small parameters appearing in (6.33) describe quite different phenomena. The parameters h/R and $h/1$ describe the initial geometry of the shell and its spatial variability, which should be supposed to be known in advance. The parameters η, $h\rho$, h/L_η and $h/1_\rho$ describe the respective predictions of orders of magnitudes of the extensional and bending strains as well as their predicted spatial variability. These parameters are not known in advance, for they strongly depend on the type of shell problem being solved, i.e. on the geometry, external surface and boundary loads, boundary conditions etc. Within the accuracy of the first approximation theory of shells it is already assumed that terms of the order of h/R, $(h/1)^2$, $(h/L_\eta)^2$, $(h/L_\rho)^2$, η and $h\rho$ can be omitted with respect to the unity. This gives us the upper bounds for estimates of various small parameters. However, in different types of shell problems the real magnitudes of some small terms may be far from their upper bounds.

For some shell problems it is possible to predict in advance the type of solution behaviour in the whole internal shell region. This prediction may then be used to compare the orders of magnitudes of various terms appearing in the set of equations (6.24), (6.30) - (6.32), what allows us to omit some terms which are of

the order of error of the first approximation theory. Then the predicted solution of the shell problem may be obtained from a considerably simplified set of intrinsic shell equations. However, it is always advisable to check at the end whether the solution calculated from the simplified equations represents indeed the predicted type of solution of the shell problem. Note that the type of shell problem is described in (6.33) by all six small parameters given above, whose orders of magnitude are entirely independent. As a result, a large variety of special cases of intrinsic shell equations may be generated from (6.24). In what follows we shortly discuss only few special cases which seem to be most important.

In the limit $b_{\alpha\beta} \to 0$ (6.24) reduce to intrinsic equations of the geometrically non-linear **theory of plates** (less error terms)

$$\{S^\beta_\alpha + A[1+\nu)S^\lambda_\alpha - \nu\delta^\lambda_\alpha S^\varkappa_\varkappa]S^\beta_\lambda - \frac{1}{2} A\delta^\beta_\alpha[(1+\nu)S^\varkappa_\lambda S^\lambda_\varkappa - \nu S^\lambda_\lambda S^\varkappa_\varkappa] +$$

$$+ D[(\rho^\beta_\alpha - \frac{1}{2} \delta^\beta_\alpha \rho^\lambda_\lambda)\rho^\varkappa_\varkappa]\}|_\beta + A[(1+\nu)S^\beta_\alpha \hat{p}_\beta - \nu S^\beta_\beta \hat{p}_\alpha] + \hat{p}_\alpha + \rho^\beta_\alpha \hat{h}_\beta = 0 ,$$

$$D\rho^\alpha_\alpha|^\beta_\beta - \rho^\beta_\alpha S^\alpha_\beta + \hat{p} + \hat{h}^\alpha|_\alpha = 0 , \tag{6.34}$$

$$\rho^\beta_\alpha|_\beta - \rho^\beta_\beta|_\alpha - \frac{1}{2} A(1+\nu)(\rho^\lambda_\alpha S^\beta_\lambda - \rho^\beta_\lambda S^\lambda_\alpha)|_\beta +$$

$$+ A\rho^\beta_\alpha S^\lambda_\lambda|_\beta + A(1+\nu)\rho^\beta_\alpha \hat{p}_\beta = 0 ,$$

$$AS^\alpha_\alpha|^\beta_\beta - \frac{1}{2} \rho^\beta_\alpha \rho^\alpha_\beta + \frac{1}{2} \rho^\alpha_\alpha \rho^\beta_\beta + A(1+\nu)\hat{p}^\alpha|_\alpha = 0 .$$

When the plate is loaded by edge forces only SIMMONDS [244,133] managed to reduce the solution of equivalent to (6.34) set of plate equations of [120] into two couple equations for the stress and deformation functions F, W , except in the case in which rotations are $O(1)$ and simultaneously the variability of deformation is very large. These extended von Karman equations are [133]

$$A(\Delta\Delta F + W|^\alpha_\beta P^\beta_\alpha) + \frac{1}{2} \langle W,W \rangle = 0 ,$$

$$\tag{6.35}$$

$$D\{\Delta\Delta W + [\frac{1}{2}(\Delta W)^2 - \langle W,W \rangle]\Delta W\} - \langle W,F \rangle = 0 ,$$

where

$$[P^{\alpha\beta} + \frac{1}{2}(1+\nu)\epsilon^{\alpha\beta}\epsilon_{\gamma\mu}F|^{\gamma}_{\lambda}W|^{\lambda\mu}]|_{\beta} = \Delta W\Delta F|^{\alpha} ,$$

(6.36)

$$\langle W,F\rangle = \epsilon_{\alpha\lambda}\epsilon^{\beta\varkappa}W|^{\alpha}_{\beta}F|^{\lambda}_{\varkappa} .$$

The (almost) **inextensional bending theory of shells** is usually defined as the one in which the extensional strains $\eta_{\alpha\beta}$ are much smaller than the bending strains $h\rho_{\alpha\beta}$, i.e. $\eta/h\rho \ll 1$. Here we assume additionally that the spatial variability of the bending strains is lower than in the general theory, $h/L_{\rho} \ll 1$. Such slowly variable bending strain states are typical for the inextensional bending deformation of the shell. If also $L_{\eta} \lesssim 1$ then within the error of the first-approximation theory (6.24) reduce to the following set of intrinsic equations of the geometrically non-linear inextensional bending theory of shells (less error terms)

$$\{S^{\beta}_{\alpha} - \frac{1}{2} D(1-\nu)(b^{\lambda}_{\alpha}\rho^{\beta}_{\lambda} - b^{\beta}_{\lambda}\rho^{\lambda}_{\alpha}) - D(b^{\beta}_{\alpha} - \rho^{\beta}_{\alpha})\rho^{\lambda}_{\lambda} +$$

$$+ D\delta^{\beta}_{\alpha}(b^{\varkappa}_{\varkappa} - \rho^{\varkappa}_{\varkappa})\rho^{\lambda}_{\lambda}\}|_{\beta} + \hat{p}_{\alpha} - (b^{\beta}_{\alpha} - \rho^{\beta}_{\alpha})\hat{h}_{\beta} = 0 ,$$

$$D\rho^{\alpha}_{\alpha}|^{\beta}_{\beta} + \underline{(b^{\beta}_{\alpha} - \rho^{\beta}_{\alpha})S^{\alpha}_{\beta}} + \hat{p} + \hat{h}^{\alpha}|_{\alpha} = 0 ,$$

(6.37)

$$(\rho^{\beta}_{\alpha} - \delta^{\beta}_{\alpha}\rho^{\lambda}_{\lambda})|_{\beta} = 0 ,$$

$$(b^{\beta}_{\alpha} - \frac{1}{2} \rho^{\beta}_{\alpha})\rho^{\alpha}_{\beta} - (b^{\alpha}_{\alpha} - \frac{1}{2} \rho^{\alpha}_{\alpha})\rho^{\beta}_{\beta} = 0 .$$

In comparison to our previous inextensional bending shell equations [185,190] derived from equivalent refined intrinsic equations the underlined terms in $(6.37)_{2}$ are taken here into account, what results from the additional requirement $h/L_{\rho} \ll 1$ used here. The presence of those therms allows for a smooth transition to the inextensional bending theory of plates, if the limit $b_{\alpha\beta} \to 0$ is taken in (6.37). The set of equations (6.37) follows also quite formally from (6.24) by taking the limit $A \to 0$, cf. [244,119].

Note that the reduced compatibility conditions $(6.37)_{3,4}$ can be solved with

respect to $\rho_{\alpha\beta}$ independently on the stress state in the shell. In this sense the inextensional bending problems of shells are geometrically determined. When $\rho_{\alpha\beta}$ are calculated, $S^{\alpha\beta}$ follow from the reduced equilibrium equations $(6.37)_{1,2}$ and then the constitutive equations (6.23) allow to recover $\eta_{\alpha\beta}$ and $G^{\alpha\beta}$.

The (almost) **membrane theory of shells** in usually defined as the one in which the bending strains are much smaller than the extensional strains, $h\rho/\eta \ll 1$. Here we assume additionally that the spatial variability of the extensional strains is lower than in the general theory, $h/L_\eta \ll 1$, what again is typical for the membrane stress states in the shell. If also $L_\rho \leq 1$ then within the error of the first-approximation theory (6.24) reduce to the following set of intrinsic equations (less error terms)

$$S^{\beta}_{\alpha}|_{\beta} + \hat{p}_{\alpha} = 0 \ ,$$

$$D\rho^{\alpha}_{\alpha}|^{\beta}_{\beta} + (b^{\beta}_{\alpha} - \rho^{\beta}_{\alpha})S^{\alpha}_{\beta} + \hat{p} + \hat{h}^{\alpha}|_{\alpha} = 0 \ ,$$

$$[\rho^{\beta}_{\alpha} - \delta^{\beta}_{\alpha}\rho^{\lambda}_{\lambda} + \tfrac{1}{2}A(1+\nu)(b^{\lambda}_{\alpha}S^{\beta}_{\lambda} - b^{\beta}_{\lambda}S^{\lambda}_{\alpha}) - Ab^{\beta}_{\alpha}S^{\lambda}_{\lambda}]|_{\beta} +$$

$$+ Ab^{\beta}_{\beta}|_{\alpha}S^{\lambda}_{\lambda} - A(1+\nu)b^{\beta}_{\alpha}\hat{p}_{\beta} = 0 \ , \tag{6.38}$$

$$AS^{\alpha}_{\alpha}|^{\beta}_{\beta} + b^{\beta}_{\alpha}\rho^{\alpha}_{\beta} - b^{\alpha}_{\alpha}\rho^{\beta}_{\beta} + A(1+\nu)\hat{p}^{\alpha}|_{\alpha} = 0 \ .$$

In comparison to our previous membrane shell equations [185,190], which followed from equivalent refined intrinsic equations, the secondary non-linear terms are omitted in $(6.38)_1$ and the underlined terms in $(6.38)_{2,4}$ are taken into account, what again results from the additional requirement $h/L_\eta \ll 1$ used here. It should be noted, in particular, that the equilibrium equations $(6.38)_{1,2}$ can not be solved here independently for $S^{\alpha\beta}$, since in $(6.38)_2$ we have the underlined terms which provide the coupling between the equilibrium equations and the compatibility conditions. As it was noted in [119], this coupling removes from the non-linear membrane theory the degeneration prevalent in the linear theory, [254]. In particular, the geometrically non-linear membrane theory of plates follows from (6.38) in the limit $b_{\alpha\beta} \to 0$. At the same time, our equations (6.38)

are considerably simpler than those which would follow formally from (6.24) by taking the limit $D \to 0$, what was suggested in [244,119].

The **bending theory of shells** equivalent to the one discussed in the section (6.1) follows from (6.24) if

$$\max \left(\frac{L_\rho}{1} \frac{h}{R} , \frac{L_\rho}{L_\eta} \frac{h}{R} \right) \ll \frac{h\rho}{\eta} \ll \min \left(\frac{1}{L_\eta} \frac{R}{h} , \frac{L_\rho}{L_\eta} \frac{R}{h} , \frac{L_\rho}{L_\eta} \frac{1}{h\rho} \right) . \tag{6.39}$$

Then (6.24) can be reduced to the set of equations (less error terms)

$$S^\beta_{\alpha|\beta} + \hat{p}_\alpha = 0 ,$$

$$D\rho^\alpha_{\alpha|\beta} + (b^\beta_\alpha - \rho^\beta_\alpha)S^\alpha_\beta + \hat{p} + \hat{h}^\alpha|_\alpha = 0 ,$$

$$(\rho^\beta_\alpha - \delta^\beta_\alpha \rho^\lambda_\lambda)|_\beta = 0 , \tag{6.40}$$

$$AS^\alpha_{\alpha|\beta} + (b^\beta_\alpha - \frac{1}{2}\rho^\beta_\alpha)\rho^\alpha_\beta - (b^\alpha_\alpha - \frac{1}{2}\rho^\alpha_\alpha)\rho^\beta_\beta + A(1+\nu)\hat{p}^\alpha|_\alpha = 0 .$$

Let us introduce the stress function F and the deformation function W by

$$S^\beta_\alpha = \varepsilon_{\alpha\lambda} \varepsilon^{\beta\rho} (F|^\lambda_\rho + \delta^\lambda_\rho KF) + P^\beta_\alpha ,$$

$$\rho^\beta_\alpha = + W|^\beta_\alpha + \delta^\beta_\alpha KW , \quad P^\beta_{\alpha|\beta} + \hat{p}_\alpha = 0 . \tag{6.41}$$

The equilibrium equations $(6.40)_1$ are approximately satisfied by $(6.41)_1$ and the compatibility conditions $(6.40)_3$ by $(6.41)_2$ provided

$$\frac{L_\eta}{1}|K|L^2_\eta \ll 1 , \quad \frac{L_\rho}{1}|K|L^2_\rho \ll 1 , \tag{6.42}$$

respectively. Then the remaining equations $(6.40)_{2,4}$ in terms of F, W take the form

$$D\Delta(\Delta W + 2KW) + \varepsilon^{\alpha\lambda}\varepsilon_{\beta\rho}(b^\beta_\alpha - W|^\beta_\alpha - \delta^\beta_\alpha KW)(F|^\rho_\lambda + \delta^\rho_\lambda KF) +$$

$$+ (b^\beta_\alpha - W|^\beta_\alpha - \delta^\beta_\alpha KW)P^\alpha_\beta + \hat{p} + \hat{h}^\alpha|_\alpha = 0 \ ,$$

(6.43)

$$A\Delta(\Delta F + 2KF) - \varepsilon^{\alpha\lambda}\varepsilon_{\beta\rho}(b^\beta_\alpha - \frac{1}{2} W|^\beta_\alpha - \frac{1}{2} \delta^\beta_\alpha KW)(W|^\rho_\lambda + \delta^\rho_\lambda KW) +$$

$$+ A[\Delta P^\alpha_\alpha - (1+\nu)P^\beta_\alpha|^\alpha_\beta] = 0 \ .$$

These are the non-linear **bending equations for shells of slowly varying curvature**, which has been proposed recently by RYCHTER [213]. Under more restrictive assumption $|K|L^2 \ll 1$, wher $L = \min(L_\eta, L_\rho, 1)$, we can also omit in (6.43) all terms with K , what leads to the non-linear **equations of quasi-shallow shells**, given by ALUMÄE [5] and KOITER [115].

The limited space of the paper does not permit to present here the explicit reduced forms of intrinsic boundary conditions to be used with each of the reduced sets of intrinsic equations discussed above. For each particular case those boundary conditions follow immediately from (6.30) - (6.32) if corresponding estimates are introduced and appropriate simplifications are made. The reader can easily derive them himself, if necessary.

Other special cases of the intrinsic shell equations and some of their applications are discussed in [51,49,190,244,119,133,18], where further references are given.

7. Closing remarks

In this report we have reviewed some achievements associated with the derivation, classification and simplification of various sets of equations of the non-linear first-approximation theory of a thin shell, deformation of which is expressible entirely by deformation of its reference surface. Basic sets of shell equations, which govern static problems of a thin shell made of a linearly-elastic homogeneous isotropic material undergoing small strains but unrestricted rotations, and associated variational principles have been formulated either in terms of displacements, or in terms of rotations and other fields or in terms of strain and/or stress measures as independent variables. References have been given primarily to those original papers and monographs which deal with general aspects of the non-linear theory of thin elastic shells and have been written in the invariant tensor notation. Apart from the unification of various partial results which are available in the literature the report contains also some original results which have not been published elswhere.

The subject of this report is quite narrow and many important aspects of the non-linear theory of shells have not been discussed. Among those associated subjects let us mention, for example, stability analysis, dynamic behaviour, large-strain theory, inelastic material behaviour, composite shells, interaction problems, higher-order shell theories, Cosserat-type theories, existence and uniqueness of solutions etc. Outside of this report are also specific problems of shells with definite geometries as well as various analytic and numerical methods of analysis of the flexible shells.

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ERRATA

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Page, place, formula of the error	Is written	Should be written
8 9th line from bottom	$\epsilon^{\alpha\beta} t_\nu$	$\epsilon^{\alpha\beta} t_\beta$
11 f. (2.16)	$c_t^2 + c$	$c_t^2 + c^2$
17 f. (3.9)$_1$	$\bar{a}_t\, d\bar{s}$	$\bar{a}_t\, ds$
17 f. (3.9)$_2$	$\sqrt{\dfrac{\bar{a}}{a}}\, \nu_\beta\, d\bar{s}$	$\sqrt{\dfrac{\bar{a}}{a}}\, \nu_\beta\, ds$
17 f. (3.9)$_3$	$\sqrt{\dfrac{\bar{a}}{a}}$	$\sqrt{\dfrac{a}{\bar{a}}}$
18 f. (3.15)$_2$	$)\cdot \delta u_j\ ,$	$)\cdot \delta u_j = 0\ ,$
21 14th and 15th line	Ω_t is the total finite	$\delta\Omega_t$ was the virtual total
22 f. (3.27)$_2$	$\boldsymbol{P}\cdot \delta\boldsymbol{u}$	$\boldsymbol{p}\cdot \delta\boldsymbol{u}$
24 7th line	[123]	[126]
29 f. (4.3)$_3$	$g = \{(h\, a^{\lambda\beta} l^{\alpha}_{\cdot\alpha})\boldsymbol{a}_\lambda + \ldots\}_{|\beta} = g^\lambda \boldsymbol{a}_\lambda$	$g = \epsilon^{\alpha\beta}\epsilon^{\lambda\mu}[(h_\mu \varphi_\alpha - h l_{\mu\alpha})\boldsymbol{a}_\lambda - h_\mu l_{\lambda\alpha}\boldsymbol{n}]_{|\beta}$
31 3rd line	[..., 244, ...]	[..., 245, ...]
35 f. (4.12)$_1$	$-\frac{1}{2}(\vartheta^\lambda_\alpha \theta_{\lambda\beta} +$	$-\frac{1}{2}(\vartheta^\lambda_\alpha \omega_{\lambda\beta} +$
36 f. (4.14)$_2$	$-\varphi^\lambda(\varphi_\alpha \varphi_{\lambda|\beta} +$	$+\varphi^\lambda(\varphi_\alpha \varphi_{\lambda|\beta} +$
39 f. (4.17)	$\bar{a}_t$	$\bar{a}_t^2$
42 6th line	[93]	[64]
42 8th line	[185, 214]	[185]
42 above f. (4.27)	[138, 198, 199]	[138, 199, 200]
44 f. (4.30)$_1$	$-[\nu_\beta f\boldsymbol{\lambda} + (\nu_\beta g + t_\beta)\bar{\boldsymbol{n}}]\bar{a}^\beta\cdot \delta\bar{\boldsymbol{r}}'$	$-\bar{a}^\beta\{[\nu_\beta f\boldsymbol{\lambda} + (\nu_\beta g + t_\beta)\bar{\boldsymbol{n}}]\cdot \delta\bar{\boldsymbol{r}}'\}$
46 9th line	[104]	[134]
46 22nd line	[130]	[133]
47 2nd line	[..., 139]	[..., 138]
48 f. (5.5)	$\boldsymbol{e} \times \boldsymbol{e}$	$\boldsymbol{e} \otimes \boldsymbol{e}$
48 7th line	[..., 73, ...]	[..., 43, ...]
50 f. (5.18)	(5.18)	(5.17)
50 f. (5.19)$_2$	$[\epsilon_{\alpha\beta}\theta_3 +$	$[\epsilon_{\beta\alpha}\theta_3 +$
50 f. (5.19)$_3$	$+ [\epsilon_{\alpha\lambda}\theta_3 +$	$+ [\epsilon_{\lambda\alpha}\theta_3 +$
50 f. (5.20)$_1$	$)\cdot \boldsymbol{a}_\lambda =$	$)\cdot \boldsymbol{a}^\lambda =$

Page, place, formula of the error	Is written	Should be written
50 f. $(5.20)_3$	$)]$.	$) + \epsilon_{\beta\lambda}(b_\alpha^\lambda \theta_3 - \frac{1}{2}\theta^\lambda \theta_{3,\alpha} + \frac{1}{2}\theta^\lambda{}_{\vert\alpha}\theta_3) - b_{\alpha\beta}(\theta_\varkappa \theta^\varkappa + \theta_3^2) + \frac{1}{2}(b_{\varkappa\lambda}\theta_\beta + b_{\rho\lambda}\theta_\alpha)\theta^\lambda]$. The expression (5.19) should still be symmetrized in $\alpha\beta$ indices.
51 f. $(5.22)_1$	$= \varepsilon_\alpha - \frac{1}{t}$	$= \varepsilon_\alpha + \frac{1}{t}$
51 f. $(5.22)_2$	$+ \frac{1}{t}\theta \ldots \quad -\frac{1}{t}\theta$	$+ \frac{1}{2}\theta \ldots \quad -\frac{1}{2}\theta$
53 f. $(5.28)_2$	$r_\alpha t^\alpha$	$r_\alpha \nu^\alpha$
56 f. $(5.38)_1$	γ_{tt}	ς_{tt}
56 f. $(5.38)_2$	$+ \frac{1}{2}(3\sigma_t - 5\varsigma_{tt})\eta_{\nu t}$	$\frac{3}{2}(\sigma_t - \varsigma_{tt})\eta_{\nu t}$
64 2nd line after 5.5.	and (5.24))	and (5.23))
66 below f. (5.76)	(5.63)	(5.62)
67 7th line	respectively.	respectively, together with one constraint condition $(5.27)_1$ multiplied by t^β.
68 f. $(5.82)_1$	$(1-\nu)(r_{\lambda\alpha}\ell^\lambda{}_{.\varsigma} + r_\alpha \varphi_\varsigma)\vert_\beta$	$\frac{1}{2}(1-\nu)\left(r_{\lambda\alpha}\ell^\lambda{}_{.\varsigma} + r_{\lambda\varsigma}\ell^\lambda{}_{.\alpha} + r_\alpha \varphi_\varsigma + r_\varsigma \varphi_\alpha\right)\vert_\beta$
68 f. $(5.82)_4$	$+ r_\varkappa \varphi_\rho)] +$	$+ r_\varkappa \varphi_\varsigma)] +$
68 f. $(5.83)_3$	$D[(\varsigma_{\nu\nu} + \varsigma_{tt})_{,\nu} + (1-\nu)\varsigma'_{\nu t}]$	$D[(\varsigma_{\nu\nu} + \nu\varsigma_{tt})_{,\nu} + 2(1-\nu)(\varsigma'_{\nu t} + \varkappa_\nu \varsigma_{\nu t}) + (1-\nu)\varkappa_t(\varsigma_{\nu\nu} - \varsigma_{tt})]$
69 3rd line	and $(5.27)_2$ as the constraints.	as the constraint.
70 2nd line below (5.86)	(5.27)	(5.19), (5.20)
70 f. $(5.86)_2$	$-\frac{1}{2}S^{\alpha\beta}[\eta_{\alpha\beta} -$	$-S^{\alpha\beta}[\eta_{\alpha\beta} -$
70 f. $(5.86)_3$	$(u - u^*) -$	$(u_i - u_i^*) -$
70 f. $(5.86)_4$	$N^\beta \nu_\beta \ldots \quad -R^+\bar{n}$	$\hat{N}^\beta \nu_\beta \ldots \quad -R^+ n$
71 below (5.87)	(5.34)	$(5.57)_1$
76 2nd line from bottom	$[92]$	$[95]$
77 17th line	$[110]$.	$[119]$.
77 19th line	distributions	measures
77 above 6.1.	(5.20).	(5.20) and (5.27)
78 f. $(6.2)_4$	$]\} +$	$]\} + k_\nu +$
78 f. $(6.2)_4$	K'_t	K'_t

Page, place, formula of the error	Is written	Should be written
78 f. $(6.2)_5$	h^3	h^2
81 f. $(6.8)_3$	x_{vt}	x_v
84 f. $(6.18)_3$	θ^2	θ^4
86 f. $(6.23)_2$	$\varrho^{\alpha\beta} -$	$\varrho^{\alpha\beta} +$
88 f. $(6.30)_5$	$]\} =$	$]\} + \hat{h}_v =$
89 f. (6.31)	$\bar{n}$	$\bar{n}_j$
89 f. $(6.32)_1$	$- A(\sigma_t - \varrho_{tt})(S_{tt} - \nu S_{vv})$	
89 f. $(6.32)_2$	$) S_{vv} - \frac{1}{2} A$	$) S_{vt} + \frac{3}{2} A$
89 f. $(6.32)_2$	$) S_{vt} +$	$) S_{vt} -$
89 f. $(6.32)_3$	$+ \frac{1}{2} A (1-\nu)$	$- \frac{1}{2} A (1+\nu)$
89 f. $(6.32)_3$	$S_{vv} + S_{tt}$	$S_{vv} - S_{tt}$
89 f. $(6.32)_4$	ϱ_{vt}	x_v
89 f. $(6.32)_5$	$N_{vv} - N_{tt}$	$S_{vv} - S_{tt}$
90 21st line	$[130]$	$[129]$
93 f. $(6.37)_{1,2}$	$\{ S_\alpha^\beta - \ldots + D\delta_\alpha^\beta (b_x^x - \varrho_x^x)\varrho_\lambda^\lambda\}_{\beta}$	$[S_\alpha^\beta - \frac{1}{2} D(1-\nu)(b_\lambda^\lambda \varrho_\alpha^\beta - b_\lambda^\beta \varrho_\alpha^\lambda)]_{\beta}$ $- D(b_\alpha^\beta - \varrho_\alpha^\beta)\varrho_\lambda^\lambda{}_{\beta} +$
98 pos. 7	(1952)	(1955)
105 pos. 148	MUSTHARI	MUSHTARI
108 pos. 200	Trands	Trends
28 below 4.1.	explicity	explicitly
33 11th line	constrains	constraints
43 below 4.5.	appear	appears
51 2nd line from bottom	η_β	η_β
53 2nd line	$\ell_{\cdot\alpha}^\lambda$	$\ell_{\cdot\beta}^\lambda$
55 f. (5.36)	$\bar{P}_{,ss}$	$\bar{P}_{,ss}$
62 f. (5.65)	(5.56)	(5.65)
80 7th line after (6.6)	was	were
82 f. (6.11)		(6.11)
19 f. $(3.18)_2$	$- \bar{b}_\lambda^\alpha M^{\alpha\beta})$	$- \bar{b}_\lambda^\alpha M^{\lambda\beta})$